Mathematical Modeling from the Teacher's Perspective

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ABSTRACT

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Applying mathematics to real world problems, mathematical modeling, has risen in priority with the adoption of the Common Core State Standards for Mathematics (National Governors Association and the Council of Chief State School Officers, 2010). Teachers are at the core of the implementation of the standards, but resources to help them teach modeling are relatively undeveloped. This multicase study explored the perspectives of teachers regarding mathematical modeling pedagogy (the modeling cycle), instructional materials, and professional collaboration, with the assumption that understanding teachers' views will assist authors, publishers, teacher educators, and administrators to develop better support for modeling instruction. A purposeful sample of six high school mathematics teachers from a variety of school settings across the country was interviewed using a semi-structured protocol. A conceptual framework developed by applying the theories of Guy Brousseau (1997) to the modeling literature guided the analysis. Qualitative methods including elements of grounded theory were used to analyze the data and synthesize the study's results. The research showed that teachers structure their instruction consistently with the modeling cycle framework, but it also uncovered the need for additional detail and structure, particularly in the initial steps when students make sense of the problem and formulate an approach. Presenting a modeling problem is particularly important and challenging, but there is inadequate guidance and support for this teaching responsibility. The study recommends the development of additional materials and training to help teachers with these steps of the modeling cycle. Furthermore, teachers find that modeling problems are engaging,

and they help students make sense of mathematical concepts. Teachers would employ modeling problems more often if they were more available and convenient to use. The study recommends that features for an online depository of modeling materials be researched and developed, including a course-based, chronological organization, a diverse variety of materials and formats, and tapping teachers to contribute their lessons.

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My six years at Teachers College have been a real treat. It is a luxury that few enjoy to launch a new career in midlife by immersing oneself in such a stimulating environment. As I reflect on the experience, memories of exciting mathematics flood back, and of a growing appreciation of the academic process. Most of all, I have deep respect for the caliber of the professors I have studied with, for their intellect, experience, generosity, and judgment. I am privileged to work with such an outstanding group.

I want to thank Bruce Vogeli, who, from our first days together in Antigua and through many courses and several publications, always encouraged my interest in the next project—and pushed me to finish the last. Phil Smith advised me on this dissertation with a trifecta of strategic advice, exacting technical feedback, and an abundance of enthusiasm and encouragement that kept me going. If I can be proud of the quality of this work it is because of him. I feel a sense of immense gratitude, and a little sadness that it is over.

I want to thank three modeling experts, Henry Pollak, Joe Malkevitch, and Rita Borromeo Ferri. Having taken multiple courses from each of them and having received their detailed advice on my research, I can say there must be no place in the world better to study mathematical modeling. I am grateful to Alexander Karp and Erica Walker for their research advice, encouragement, and their roles on my committees. I would also like to thank Sandra Okita and Patrick Gallagher for their thoughtful suggestions and service on my dissertation defense. Victoria Marsick has been an uncanny presence at key points in my time at Teachers College. Her workshop at an open house event seven years ago drew me to enroll. Her methods course filled a crucial gap in my experience when I needed it for this thesis. I am grateful for both, and I hope we keep crossing paths at just the right moments.

Many thanks go to my Modeling Handbook coauthors, Andrew Sanfratello and Luke Rawlings, for their collaboration. I am also grateful to several mathematics and teacher friends for their advice and support: Allison Curran, Tammy Applegate, Lisa Babinet, Wendy Qui, Pat Brent, Andrew O’Grady, and Chris Cornwell.

Finally, I would like to thank my wife and children, Nan, Andrew, and Grace, for their patience, support, and love while I have worked on this project, and always.

Chapter I: Need for the Study, Purpose, and Procedures

Need for the Study

This is an exciting time in mathematics education. The Common Core State Standards for Mathematics (Common Core) have been adopted but not yet implemented. Proponents of teaching the application of mathematics to real world problems have won the inclusion of mathematical modeling in the standards. Fulfilling the promise of this policy change—successfully infusing modeling instruction into mathematics classrooms—is challenging for several reasons: modeling is difficult, teachers generally have little experience with modeling, and there are limited modeling materials in current textbooks. This study aims to collect and analyze teacher feedback on three forms of assistance policy makers, publishers, and teacher educators are developing for mathematical modeling.

The first is a theoretical framework of modeling, the "modeling cycle." From the early arguments for mathematical applications fifty years ago (Bell, 1971; Freudenthal, 1968; Pollak, 1968, 1969), through several decades of research and development of modeling in the curriculum (Burkhardt & Pollak, 2006; Garfunkel & Malkevitch, 1994; Lesh & Yoon, 2007; Niss, 2012; Usiskin, 1997), authors have defined and discussed modeling in terms of an iterative cycle of the steps involved. Some versions of the modeling cycle have been explicitly targeted for teacher use or for classroom instruction of students (Blum & Borromeo Ferri, 2009; Blum, Niss, & Galbraith, 2007), and the Common Core authors make prominent use of the modeling cycle in their instructions to teachers (National Governors Association and the Council of Chief State School Officers [NGA & CCSSO], 2010). Yet in practice many if not most modeling lessons do not follow the modeling cycle nor explicitly mention it. We need to understand this inconsistency if this central element of the policy-making and research communities is to have maximum benefit.

Crucially, the actual implementation of mathematical modeling falls to teachers in the classroom. Success depends on their mathematical abilities to model, their pedagogical skills to teach it, and their decisions to allocate time and priority to it. Teacher perspectives and knowledge regarding the modeling cycle and other theoretical aids, as well as forms of practical assistance offered to them, are therefore central to whether these supports are employed, whether they are useful, and in the end whether students learn modeling. We need to judge these aids, in part, by teacher feedback, and we need to understand teacher perspectives to do so.

A second support for teachers is mathematical modeling instructional materials, lesson plans, and assessments provided by third parties. All but the newest textbooks do not yet cover the Common Core standards, so teachers need to (and are free to) select modeling materials from a variety of third parties based on their own criteria and judgment. Modeling's characteristic reliance on a real world problem context (which teachers can introduce to students through authentic online sources in some cases) and the fragmentation of educational sources driven by the technological revolution are two more factors that put teachers in control of modeling instructional materials. We need to understand teachers' requirements and preferences in order to provide them with the best possible materials.

Finally, many teachers are not confident teaching mathematical modeling for the first time without professional assistance (Gould, 2013). The variety of potential professional development, training, departmental study, mentoring, push-in specialists, assessment norming, and similar collaborative activities is the third area of support for mathematical modeling investigated in this study. Again, teacher judgments are probably one of the best measures of the relative merits of these alternatives, and they are a key determinant of teacher participation. Gathering and understanding teacher perspectives regarding forms of collaboration in the teaching of mathematical modeling is this study's third focus.

Purpose of the Study

The purpose of this research was to study the implementation of the Common Core modeling standard by applying the theories of Guy Brousseau to explore teachers' beliefs about and instructional practices related to mathematical modeling.

Questions examined in this study were as follows:

1. How well does Brousseau's theory of didactical situations align with our current understanding of the modeling cycle?

2. To what extent do teachers consider the modeling cycle as an important framework to structure their instruction? How, if at all, do they report it influences the way they prepare to teach mathematical modeling and assess students?

3. What are teachers' perceptions of the appropriateness, ease, and usefulness of mathematical modeling lessons? What additional resources, if any, do teachers report as necessary for the teaching of mathematical modeling?

4. In what collaborative activities do teachers engage while planning, implementing, and evaluating mathematical modeling lessons? What additional forms of collaboration, if any, do teachers report they would participate in if they were available?

Procedures of the Study

The study employed a qualitative approach (Charmaz, 2014; Glaser, 1978; Glaser & Strauss, 2009; Merriam, 1998; Yin, 2009), interviewing teachers to gather their perspectives. In preparation, I performed a preliminary review of the research on teaching mathematical modeling—particularly regarding the modeling cycle, instructional materials, and professional collaboration—and the works of Guy Brousseau (1997). I applied Brousseau's theory of didactical situations in mathematics to the teaching of modeling and derived an initial set of questions to discuss with teachers. Further study of the literature continued in parallel with the development of the interview topics and their exploration with teachers. Thus the activities of using and extending Brousseau's theory and that of gathering and interpreting teacher perspectives informed each other and developed together. A final synthesis of the perspectives framed by an extended theory constituted the results of the study.

Subjects of the study. I solicited six high school mathematics teachers who had taught mathematical modeling and who agreed to share their perspectives with me in one or more meetings. A set of subjects for the study was desired that would be broadly representative of the Common Core modeling standard's mandate and be motivated to discuss their perspectives with me. I invited teachers I knew personally or professionally to participate in the study. I also attempted to interview teachers from a variety of schools: the low income public school where I work, other urban public and private schools, and suburban schools. A sample size of six teachers was large enough to gather a variety of opinions while being a practical size to analyze carefully.

Prepared materials. To facilitate discussions with teachers, materials pertinent to each of the research questions were reviewed in the interviews. The documents were drawn from three sources: publicly available standards from the Common Core (NGA & CCSSO, 2010); modeling materials from coursework at Teachers College (DePeau, 2012; Tan, 2012); and the participants' own lesson plans and examples of student work. I selected the materials based on three criteria: that they were of high quality or widely used, that they highlighted one of the features explored by the research questions (i.e. an explicit modeling framework, a supplemental resource, or support for collaborative activity), and that they were relevant to the teacher's work.

Two of the modeling lessons discussed with the study participants were part of a project related to this study: modeling chapters for a book coauthored with Andrew Sanfratello and Luke Rawlings, the COMAP Mathematical Modeling Handbook III (Huson, 2015). Handbooks I and II were 26 modeling lessons and matching assessments, respectively. Handbook III continued with a subset of those lessons, developing "paradigms" to assist teachers to implement modeling in their classrooms. Thus for a particular two-day lesson, Handbook I contains student handouts, Handbook II contains assessments, and Handbook III advises on strategies for teachers. This study's research questions were themes for my lesson paradigms (modeling cycle, outside resources, and collaboration), and Brousseau's theory informed the work.

Interview protocol. The meetings with teachers followed a protocol for efficiency and to ensure the research questions were covered. I prepared a background letter, the modeling documents discussed in the previous paragraphs, and a set of interview questions based in part on Brousseau's theories. I also provided the subjects with an IRB disclosure form and a short survey to collect data including the courses they teach, their teaching experience and history, their knowledge of modeling, and similar background information. Each interview lasted roughly an hour. The interviews were recorded and transcribed, and then the teachers' perspectives were analyzed to answer the research questions.

The interviews were a free-flowing discussion, but were guided by a written protocol to help ensure that data addressing all of the research questions were gathered. The general pattern was to introduce each of the research topics and then to ask the teacher to compare and comment on example documents, first considering the teacher's own materials. Over the course of the data gathering, incremental changes were made to the protocol based on results from previous interviews and the ongoing literature review.

Conceptual framework. To aid in the data collection and analysis a conceptual framework was developed and employed based on Guy Brousseau's theory of didactical situations in mathematics (1970/1997). Brousseau is a prominent French educator and researcher. The International Commission on Mathematical Instruction (ICMI) awarded Brousseau its inaugural Felix Klein award in 2003 for his lifetime of achievement.

This study's focus on the modeling cycle and instructional materials aligns well with Brousseau's ideas, and key components of his theory illuminate facets of teaching modeling. His contrast of *adidactical* versus *didactical* problems is similar to modeling's goal to place real world problems in a classroom environment. The unusual demands we require of students as they model illustrate renegotiating what he calls the *didactical contract*. His term *milieu* refers to the learning environment encompassing the student, teacher, instructional materials, and problem—which with online resources may extend well beyond the classroom to the real world itself. His explanations of the actions and intentions of teachers frame the study's approach to gather teacher perspectives and their analysis. Employing Brousseau's theories to study the Common Core modeling rollout promises to help us both understand how to meet our classroom goals and extend his work to a new application and context.

Analysis. The interview transcripts were coded and analyzed using qualitative methods (Creswell, 2013; Merriam, 1998; Yin, 2009) with elements of grounded theory (Charmaz, 2014; Glaser & Strauss, 2009). Together with the lessons and student work provided by the teachers, the interview data were summarized and analyzed to answer the research questions. I followed an established policy to safeguard confidential materials and return or dispose of them when the research is complete.

Chapter II: Literature Review

To what extent are Brousseau's theories applicable to modeling, as understood from the Common Core modeling standard (NGA & CCSSO, 2010) and modeling research that it is based on? This literature review first summarizes the history and main ideas of the theory of didactical situations and the modeling research, and it then maps the connections between the two bodies of work. Finally, it explores how findings from the theory of situations with no analogies in Common Core may shed new light on the teaching of modeling.

The structure of this section is as follows: an overview of the major areas of the two sets of theories with introductory explanation of terms, historical background on the development of Brousseau's work and the modeling literature, a detailed comparison of the two schools' views on each of the major areas of overlap, an exploration of application of the theory of didactical situations to modeling topics that may be new or without existing analog. As a reminder of the process used to select and interpret the literature reviewed in this research, which is explained in detail in the methods chapter, the subjects' interview responses guided the literature review in keeping with the grounded theory methodology (Charmaz, 2014). Some teacher perspectives are briefly mentioned as context in the literature review, but the bulk of those data are presented in the results chapter.

Relevance of the Theory of Didactical Situations to the Research Questions

Guy Brousseau is a prominent French mathematics teacher and researcher who developed an extensive pedagogical theory he calls the theory of didactical situations. Brousseau was originally considered as a component of this study for two reasons. First, his theory is comprehensive, coherent, and rich with interesting theoretical concepts. His abstract perspective complements the practical bent of most of the literature on modeling. Second, while his work is recognized in Europe, his research is not well known in the U.S., in part because only some of it has been translated to English. In his introduction to one such translation Kilpatrick praises the body of research but laments that it "has as yet had only modest influence in the Anglophone world" (McShane Warfield, 2014, p. v). Therefore applying Brousseau's theory to study modeling is an opportunity to address a gap in the English-based literature.

Before committing to Brousseau's theory, an initial read of the primary English translation of his books (1997) was made to test its suitability for this study. Brousseau's student-centered and constructivist approach—further discussed below—appeared practical and applicable, and many of the concepts of his theory of didactical situations promised to have potential relevance to teaching mathematical modeling. In grounded theory one use of the literature is "to derive a list of questions that you want to ask your respondents or that guide your initial observations" (Strauss & Corbin, 1990, p. 50). This was exactly the benefit of the initial review. The interview protocol that was approved by the Teachers College IRB and used for the first three interviews had several questions exploring concepts from the theory of didactical situations. Also, the publication of the four modeling chapters (Huson, 2015) I composed for this study included a preamble informed by Brousseau's work. In particular, Brousseau's concept of the *milieu*, the learning environment broadly defined, and the *didactic contract*, the implicit agreement between student and teacher, helped define themes that ran through the four modeling handbook chapters.

Brousseau's work thus shown in initial practice to be a useful theoretical base for the study, a more thorough review was made for concepts and relationships that would apply to teaching mathematical modeling. The results are in the following pages. They are preceded with a short biography of Brousseau for background, and then organized as a list of topics or terms.

Each concept, often colorfully named, is defined as per Brousseau's thinking and then extended to the mathematical modeling context. The resulting questions posed to the study's subjects follow, sometimes with a description of their evolution as the discussions with teachers give further meaning to the theory's concepts. Thus, following the grounded theory tradition, one of the study's goals is to relate the theory growing out of the views expressed by the study's subjects to the existing literature in a cumulative and integrated fashion (1990).

The conceptual framework for this study is based on Guy Brousseau's theory of didactical situations in mathematics. The significance of this theory should be appreciated, notwithstanding its lack of familiarity in the English-speaking community. It is built on an enormous body of experimental evidence collected through the efforts of many teachers and researchers, under ideal conditions, over a period of decades (Brousseau, 1997). Jeremy Kilpatrick says it "represents an extensive share of the best thinking we have concerning how mathematics is and can be taught" (McShane Warfield, 2014, p. 6), and Alan Schoenfeld suggests Brousseau's work as a model, recommending that "a research theory as comprehensive as the didactique could be developed based on English-speaking country's traditions" (Schoenfeld, 2012, p. 590). The International Commission on Mathematical Instruction awarded its inaugural Felix Klein Medal to Brousseau for a lifetime of achievement. Their citation:

> This distinction recognizes the essential contribution Guy Brousseau has given to the development of mathematics education as a scientific field of research, through his theoretical and experimental work over four decades, and to the sustained effort he has made throughout his professional life to apply the fruits of his research to the mathematics education of both students and teachers. … [His] theory, visionary in its

integration of epistemological, cognitive and social dimensions, has been a constant source of inspiration for many researchers throughout the world.[1]

A brief outline of Brousseau's teaching career and early research follows (for more detail in an English language source, see Brousseau, 1997, pp. xv-xix). Brousseau was born in 1933 and began as an elementary school teacher in 1953. During that year he married Nadine Labeque, also a teacher. The two taught at the same school in their early years, and she became a key collaborator in research and experimentation throughout his career. Brousseau's ideas developed and his role in the educational community grew during the 1960s with milestones including his first publication in 1965 (a manual for teachers), a proposal for a research program in mathematics teaching (at a 1968 colloquium), and the first presentation of the theory of didactical situations in 1970 (Brousseau, 1997).

It is perhaps noteworthy to this modeling study that in addition to his duties as a teacher, in 1961 Brousseau worked with local farmers to apply mathematics to optimize their production, thus gaining practical experience in real mathematical modeling (Brousseau, 1997). Brousseau addressed issues related to teaching mathematical modeling in later publications, which are discussed later in this section.

In the 1960s constructivism as proposed by Piaget rose to prominence: that students construct knowledge rather than absorb it. Brousseau set out to substantiate the theory with experimental evidence and to study the conditions under which students would best learn and construct mathematical understanding. Brousseau also recognized the importance of social interplay among students learning mathematics and the community context of mathematical practice. He credits Piaget and Vygotsky as leading him to focus on psychological studies of

[1] http://www.mathunion.org/icmi/other-activities/awards/past-recipients/the-felix-klein-medal-for-2003/

children learning mathematics within a context “at once material, social, and cultural” (Brousseau 1999, p. 69).

Brousseau’s theory of situations arose within the consensus popular in the 1960s that school mathematics should reflect the way mathematicians practice their craft. The streamlined final form of professional results suggested that students should be presented with well organized, axiomatically-based content in a linear sequence. However, mathematics does not really originate in such a clean fashion. Contextual factors are present initially when mathematicians innovate, and Brousseau believed that students too would best construct their own understanding of mathematics through participation in complex, problem solving situations (Brousseau, Brousseau, & Warfield, 2013). The study of such learning situations is the core of his research: well chosen problems that stimulate the birth of mathematical concepts. In Brousseau’s words, these problems or situations “are many-faceted adventures that pull together a whole conglomeration of connaissances that will be provoked, activated, invented, used, modified, and verified, around a project of a mathematical nature dealing with an essential mathematical notion” (2013, p. 131).

Professional mathematicians practice in a community and progress requires that ideas are shared and debated. So too, classrooms are communities, and Brousseau was perhaps most innovative in the use of social factors in problem solving. His mathematics problems are often structured as games, setting individuals and teams in collaboration and competition. His students communicate, debate, offer evidence, prove. The term *milieu* represents the social structure that is intentionally combined with a mathematics problem to compose a learning situation. Together the problem and social components trigger the construction of knowledge within a student. “The term ‘situation’ designates the set of circumstances in which the student find herself, the relationships that unify her with her milieu, the set of ‘givens’ that characterize an action or an evolution. ... [it] necessitates an adaptation, a response, by the student” (Brousseau 1997, p. 214). Understanding grows communally and the meaning of a particular mathematical concept is derived from its usefulness to solve, communicate, or debate a problem (Brousseau et al., 2013).

The third contribution of Brousseau’s theory, in addition to insight into problem situations and the classroom milieu, regards the teacher’s function. Teachers and students accept certain responsibilities in school. Their understanding of each other’s roles may be implicit but it is quite concrete, to the point of justifying the term contract, what Brousseau calls a *didactic contract*. The theory of situations identifies key actions the teacher must take, and highlights a central paradox in his role. On the one hand, he cannot simply teach the mathematics. Since meaning must be constructed by the learner, he must convince the student to find the mathematics in the logic of the problem. Brousseau says the teacher must *devolve* responsibility to the student. Successful devolution requires that the teacher both maintain direction of the lesson and force responsibility on the students. Additionally, as the students’ initial notions are formulated and refined through group use and discussion, they must be standardized according to the norms used in adult practice and connected within the overall framework of mathematics. Brousseau says that student knowledge must be *institutionalized*.

In summary, Brousseau’s goal was a theory of optimal learning situations, milieu, and teacher’s functions in terms of the didactic contract, devolution, and institutionalization, supported by experiment (Brousseau, 1997). A research center was established, the COREM (Center for Observation and Research on Mathematics Teaching), and with it in 1972 the Jules Michelet school at Talence, France, a laboratory school where real classes could be modified and observed. The Michelet school had extra teachers and was equipped for videotaping. In this setting over roughly 25 years, Brousseau and other researchers (among them Nadine Brousseau having a central role) developed and tested the ideas that compose the theory of didactic situations (Brousseau, 1997).

Brousseau's writings are detailed discussions of experiments carried out at the Michelet school. Over the years elementary school-level mathematical content areas were studied ranging from division and multiplication, to probability and statistics, to fractions and decimals. The work involved developing a series of lessons using the methods of the theory of situations, preparing and practicing their presentation, then observing their use with real children. After review the lessons would be revised and taught the following year. To give an idea of the scale of the research undertaken, each lesson in the decimals unit required 30 hours of work before its use (Brousseau, 1997, p. 117). That particular material was reworked and taught for ten consecutive years from 1976 to 1986.

Much of the research materials are available publicly for review and continued research, including detailed lesson descriptions, videotapes of the classroom. The material is hosted by the Jaume 1 University of Castillón de la Plana, Spain, and Brousseau has preserved more writings on an archival website, part of which has been translated and is mirrored in an English language version by McShane Warfield (2014).

The History of Mathematical Modeling in Education

In the same timeframe that Brousseau launched and developed his research, the teaching of mathematical modeling also enjoyed steady progress, eventually leading in the United States to the inclusion of modeling in the Common Core standards. In this section I give a brief history of school modeling, beginning with Pollak and Freudenthal in the late 1960s, then through the association with problem solving research in the following decades, and finally the evolution of modeling standards from the National Council of Teachers of Mathematics (NCTM; 1989, 2000) and the Common Core (NGA & CCSSO, 2010).

Solving real world problems has obviously been a part of mathematics since its inception, but I will begin this history in the late 1960s with a conference Hans Freudenthal organized to promote applications and modeling in education, in part as a reaction to the theoretical emphasis of the "new math" movement dominant at the time. Freudenthal (1968) opened the conference with a talk titled, "Why to teach mathematics so as to be useful." He claimed, "In its first principles, mathematics means mathematizing reality" (1968, p. 7). Henry Pollak, speaking at the same conference, continued by identifying misconceptions and challenges that stood in the way of teaching modeling more effectively and by proposing a number of remedies, including a library of modeling problems. In his closing he said, "The big unfinished task is to collect appropriate examples of honest applied mathematics for earlier levels of education. I know it is possible to bring real applications into the secondary, and even the elementary, school and to motivate and illustrate much mathematics by such examples" (1968, p. 31). With such calls, modeling education began what Niss, Blum, and Galbraith called the "advocacy phase" (2007, p. 28).

Bell concurred that instructional materials were central to moving modeling into the curriculum. "I believe that the main difficulty is that not enough attention has been given to the production and classroom trial of material that would make mathematical models and applications a usable part of the existing school courses" (1971, p. 249). Bell worked at the University of Chicago School Mathematics Project with Usiskin on one of the efforts to design new curriculum, curriculum in part built around a much larger role for applications and mathematical modeling. The pedagogical design was to order a sequence of modeling problems to develop a particular mathematical concept. The “application sequences are grounded in fundamental meanings of one or more arithmetic operations” (Usiskin, 1997, p. 75). The meaning-making function of problems, arranged in a well-designed sequence, was thus a central theme at the School Mathematics Project at around the same time as it was in Brousseau’s COREM research institute.

Problem solving became a central goal for mathematics instruction in the 1980s. The NCTM featured it prominently in its 1980 Yearbook (Krulik & Reys, 1980) and a decade later in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Mathematical problem solving and mathematical modeling have many similarities, and support for them in education grew in parallel. Modeling tends to be more closely connected to the real world problem context, and it places a greater emphasis on an iterative or cyclical process to develop solutions (Schoenfeld, 1992; Zawojewski, 2010). Through the 1980s and 1990s modeling curricula were written (Burkhardt, 1983, 1989; Garfunkel & Malkevitch, 1994), and an associated research community developed. The International Conferences on the Teaching of Mathematical Modelling and Applications (ICTMA) was founded and began biennial conferences in 1983 (Blum & Niss, 1991; Lesh, Doerr, Carmona & Hjalmarson, 2003; Niss et al., 2007).

In terms of education policy in the United States, the groundwork for the inclusion of modeling in the high school curriculum was laid by the NCTM’s Principles and Standards for School Mathematics (2000), and then it became the official policy of the majority of states with the adoption of the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). The Common Core notwithstanding, it should pointed out that modeling is not actually being taught in classrooms to the degree called for by the standards. In 2007, for example, Niss, Blum and Galbraith noted that there remained a "substantial gap" in modeling educational practices, indeed that "genuine modelling activities are still rather rare in mathematics classrooms" (2007, p. xi). In a recent lead article of the most prominent publication for high school mathematics teachers, *Mathematics Teacher*, ten veteran educational leaders called modeling "a central goal of high school instruction," but warned that, "We have a long way to go before those recommendations are common practice in most U.S. high schools" (Fey et al., 2014, p. 489).

The Teaching and Learning of Mathematical Modeling

The research literature on modeling is extensive. This section surveys major areas of investigation and references. It begins with the nature of modeling as a mathematical activity, then distinguishes it from a related area, mathematical problem solving, and finishes with a discussion of research into how modeling should be taught. Readers interested in more background on the modeling literature are recommended to consult Legé (2003).

The nature of mathematical modeling. Mathematical modeling is the application of mathematics to a real world situation or problem. This study uses the Common Core's definition: "Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (NGA & CCSSO, 2010, p. 72). Modeling researchers and curricula developers (Blum & Niss, 1991; Borromeo Ferri, 2006; Burkhardt & Pollak, 2006; Lesh & Yoon, 2007; Niss, 2012) vary in their emphasis and precise definitions of modeling but they consistently bifurcate the process between mathematical elements and extra-mathematical or real world components (see Figure 1).

The activities that distinguish modeling are those steps relating the two realms, when mathematical ideas are used to represent extra-mathematical elements or relationships. For

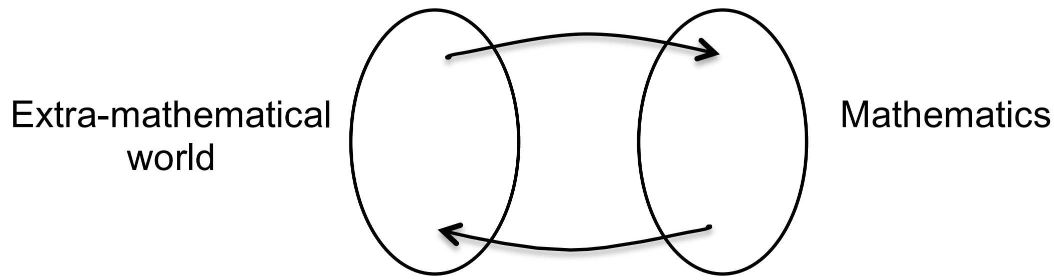

Figure 1: Activities in modeling relate the interplay of mathematics and the real world (Niss, Blum, & Galbraith, 2007, p. 4).

example, in the early steps of solving a problem, judgments must be made as to the salient features of the real world situation and how to represent them mathematically. In this way, elements of the situation are "mathematized" or "decontextualized." The Common Core uses the expression that a mathematical model is "formulated" based on the real world problem. This mathematical representation of the problem is then analyzed or solved using traditional mathematical techniques. A second stage of transition between mathematical and extra-mathematical realms ensues as the mathematical solution is interpreted within the real world context and evaluated as to its suitability. In practice the initial results are often found lacking, and improvements must be made by returning to the assumptions and simplifications, refining the mathematization, and re-solving the mathematical model. Thus modeling is often an iterative process, and the term "modeling cycle" is used to describe these repeated steps or phases. Modeling cycle representations vary among researchers, but there are always two transitions between extra-mathematical and mathematical contexts, first as the real world situation is mathematized and later as the mathematical solution is interpreted and evaluated in the context of the problem situation. Some depictions of the modeling cycle have been designed for use with teachers or in classrooms (Blum & Borromeo Ferri, 2009; Blum, Niss, & Galbraith, 2007). For this study, the Common Core's version of the modeling cycle was used, as is discussed in the Common Core Modeling Standard section below.

Relevant findings from research on problem solving. Mathematical problem solving is an area of research and policy that is closely associated with modeling and that has received a great deal of attention (Blum & Niss, 1991; Niss et al., 2007; Schoenfeld, 1992; Usiskin, 1997; Zawojewski, 2010). In the following paragraphs the relationship between problem solving and modeling is explained, and then findings from research on problem solving that may be relevant to modeling are discussed.

Schoenfeld is a prominent researcher in mathematical problem solving (1992). He explained the relationship of modeling to problem solving in a 2013 address at Teachers College, Columbia University. Both are involved with using mathematics to make sense of a situation. "The purpose of mathematics education has always been to help students to do mathematical sense making" (Schoenfeld, 2013, p. 13). The distinction is regarding the subject of investigation, whether a mathematical situation is examined or one from the real world. What Schoenfeld calls traditional problem solving can overlap with what he terms applied problem solving (modeling). The situation is depicted by the diagram in Figure 2.

Given modeling's commonality and overlap with problem solving, the large quantity of research into teaching problem solving offers a guide to that of modeling. Lester (1994) summarized the findings from 25 years of research into teaching problem solving, which had received a great deal of study in the two decades preceding his review. The consensus was that in order to become proficient students must solve many problems over a long period in well-planned and highly prioritized instruction, and that "teaching students *about* problem-solving

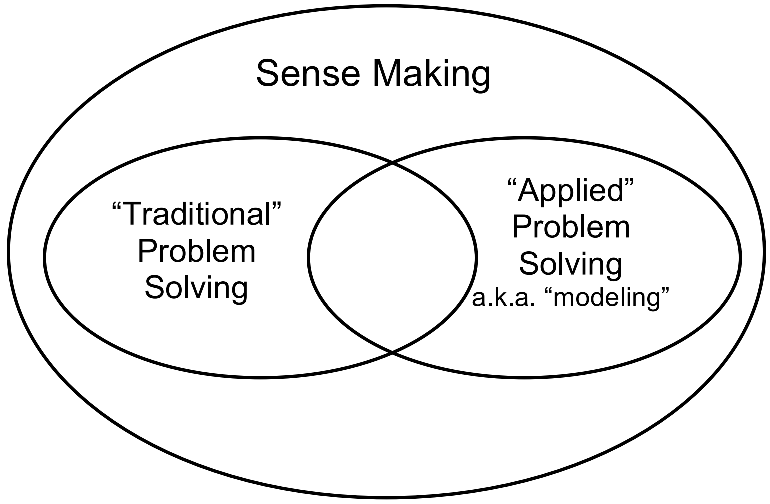

Figure 2: Schoenfeld's diagram of the relationship of modeling to traditional problem solving. Both employ mathematics to make sense of a situation (2013, p. 15).

strategies and heuristics and phases of problem solving … does little to improve students' ability to solve mathematics problems in general" (Lester, 1994, p. 666). Additionally, in his recommendations for more research to guide instruction, he prioritized three areas: the role of the teacher, classroom social interaction, and examining student groups as the unit of study. For the purposes of this study, it is interesting to note the overlap of Lester's research priorities with several features of Brousseau's theory of situations: the teacher's role in terms of the phenomena of devolution and the didactic contract, and the construct of the milieu encompassing classroom social features (Brousseau, 1997).

Types and features of problems in modeling. As in the case of problem solving, at its core mathematical modeling involves a non-routine mathematical task, in modeling one prompted by a real world problem situation. The characteristics of a "good" modeling problem have received considerable study. Authenticity is a quality that disposes students to accept and productively engage with a problem (Galbraith, 2007; Niss et al., 2007; Vos, 2011). Taking another tack, Niss, Blum, and Galbraith categorized tasks as word problems, standard applications, or modeling problems based on the extent to which they "genuinely meet real-world criteria" (2007, p. 11), which means that, in practice, they demand the entire modeling cycle in their solution. Pollak said that we should employ "honest" problems (1969, p. 401), meaning that the student must understand the relationship between the real world situation and the mathematical model, be able to derive the model including necessary assumptions and approximations, and have a basis for interpretation and validation by real world criteria. Yet another trait is what Julie called "usability" (2002, p. 4), a student's sense that a problem might actually pertain to his or her life.

How modeling problems are generated and used has also received considerable study (Galbraith, Stillman, & Brown, 2010). Galbraith (2007) highlighted the distinction between modeling problems that serve to introduce a particular content area of mathematics and problems suitable to teach modeling for its own sake. Targeting a desired mathematical concept is called "modeling as vehicle," and such use is the dominant form for modeling problems in schools (Julie, 2002). Less commonly, teaching "mathematical modelling as content entails the construction of mathematical models for natural and social phenomena without the prescription that certain mathematical concepts or procedures should be the outcome of the model-building process" (Julie, 2002, p. 3). According to proponents of teaching modeling as content, the dominance of practicing modeling as vehicle undermines the development in students of an orientation to analyze the world mathematically. Authenticity is diminished if every problem encountered in school is contrived for a content-teaching purpose. Instead, Julie recommends that skills specific to modeling should be targeted, and a mathematical orientation to looking at the world should be cultivated. Those skills, or competencies, most important to modeling are discussed in the next section.

Modeling competencies. A comparison with problem solving shows that the open-ended character of modeling demands of the student additional capabilities, or what Blomhøj and Jensen call modeling competencies (2003). In the case of problem solving, the given situation may be abstract or practical, but it is well structured and complete. For modeling, however, the real world situation that is the subject of investigation is not so neatly contained. There may be missing information that must be assumed or estimated, and there will probably be extraneous information that must be ignored. Moreover, the suitability of a solution to a modeling situation must be evaluated based on real world criteria. This often requires an element of experience or judgment. Teaching the competencies involved in modeling takes longer than teaching routine problems, and it is more difficult for teachers (Burkhardt, 2014; Niss et al., 2007).

Blomhøj and Jensen argued that the complexities of modeling competencies suggested two alternative teaching strategies: the "holistic approach" and the "atomistic approach" (2003, p. 128). The holistic approach entails working on complete modeling tasks with the goal of developing overall competency and the additional benefit of the motivation that comes from working on authentic problems. Unfortunately, this is a very time-consuming approach. The atomistic approach focuses on the mathematization step by using relatively well constructed problem situations and limiting the complexity of the real world context and evaluation. Blomhøj and Jensen recommended that a balance be found between the holistic and atomistic approaches, but they also warn, "Special attention must be paid to the inadequacy of the atomistic approach since this is tempting to adopt due to its conformity with traditional teaching strategies in mathematics education" (2003, p. 137). The Common Core modeling standard attempts to strike such a balance. The standard is discussed in the next section.

The Common Core Modeling Standard

The modeling cycle. The modeling cycle is a term used throughout this study and in the literature on mathematical modeling. This section defines the term and provides background to explain the Common Core modeling standard (NGA & CCSSO, 2010). This study uses the Common Core's definition of the modeling cycle because of the policy's wide adoption across the United States educational system. Indeed, the standard represents the culmination of modeling researchers' and educators' efforts to promote and develop a consensus for modeling instruction.

The Common Core defines modeling as follows: "Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (NGA & CCSSO, 2010, p. 72). The term modeling cycle refers to the overall process of making and using a model. It can be thought of as a series of steps, as shown by the flow chart in Figure 3.

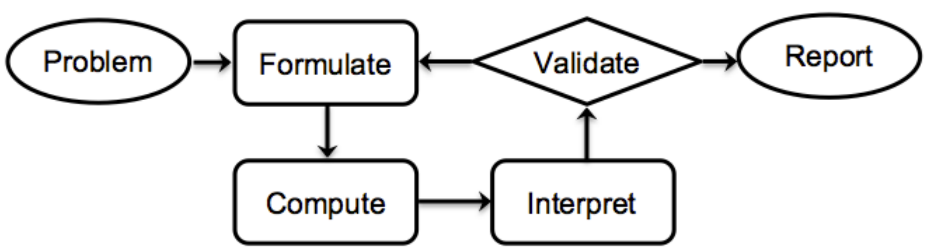

Figure 3: The modeling cycle as it is depicted in the Common Core State Standards for Mathematics (NGA & CCSSO, 2010).

In this study the six modeling steps are generally referred to by the names in the diagram, and I use the definitions of the Common Core standards:

> [Problem:] identifying variables in the situation and selecting those that represent essential features,
> [Formulate:] formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables,
> [Compute:] analyzing and performing operations on these relationships to draw conclusions,
> [Interpret:] interpreting the results of the mathematics in terms of the original situation,
> [Validate:] validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
> [Report:] reporting on the conclusions and the reasoning behind them. (NGA & CCSSO, 2010, p. 72)

The cycle is not necessarily meant to suggest a linear process, and it certainly is not meant that a model should be built by mechanically following a list of steps. In fact, the process should be fluid and holistic: “Choices, assumptions, and approximations are present throughout this cycle” (NGA & CCSSO, 2010, p. 73). Similarly, in this study’s research questions and in the discussions with teachers, the term modeling cycle is employed in an expansive sense to mean the processes and abilities the Common Core describes across the overall modeling standard.

Content and practice standards. The Common Core State Standards for Mathematics (NGA & CCSSO, 2010) is structured as two related parts: content standards listing what students should be able to do and practice standards defining how they should go about it. Mathematical modeling is the only Common Core standard that is both a content standard and a practice standard. That is, students need to learn to model (a content standard), but they should also be able to apply a broad range of mathematics to engage with the real world (a practice standard). The practice standard states, “Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace” (p. 7). The Common Core’s listing of mathematics content areas includes frequent references to employing those mathematical techniques to build models. For example, the content standard relating to mathematical functions states that students should be able to “construct and compare linear, quadratic, and exponential models and solve problems” (2010, p. 70). Thus, the goal to teach students to apply mathematics “to everyday life, work, and decision-making” (2010, p. 72)—modeling—figures prominently across the entire Common Core standards for mathematics.

Guy Brousseau and the Theory of Didactical Situations

In interviews, teachers say students learn best by discovery, rich classroom interaction motivates students, and a well-designed modeling problem makes mathematics meaningful. This literature review of Brousseau’s theory and the research into the teaching of mathematical modeling frame those teacher perspectives. The advantage of Brousseau’s theory is its breadth and detail built through decades of development and experimental verification. To evaluate his theory vis-a-vis modeling research and the Common Core policy, and to use it as the framework to understand teacher perspectives regarding the teaching of modeling, this section first summarizes the theory of didactical situations and then juxtaposes its main elements with analogs in the modeling research. The core of the theory is discussed first: the student’s construction of meaning in a social setting with important roles taken by the teacher. Secondly, the sequencing of progressive types of lessons and the development of mathematical concepts is described. Obstacles, paradoxes, and instructional fallacies that Brousseau discovered are categorized next. The limitations of Brousseau's research, in particular his call for didactic engineering (as opposed to research) and the practical development of materials for use in schools is then discussed. Finally, Brousseau's comments on mathematical modeling finish the section.

Constructivist basis, meaning making, understanding. The foundation of the theory of didactical situations is epistemological: How does someone know something? A student knows a mathematical concept because he can solve a certain type of problem that requires the concept or procedure. We cannot verify it directly, but we imply the existence of this knowledge because we observe decisions the student makes, behaviors, interaction with the problem, peers, and the teacher. How does the student know it is true? The teacher may tell him it is true, what Brousseau labels "classical" instruction. It is straightforward to teach this way but limits understanding. The mathematics' significance, in that case, is that it satisfies the teacher. The student is forever looking for clues from the teacher, never linking concepts into a connected mathematical framework. The justification of the fact is "cultural" in the sense that it appears to be a classroom convention (Bosch & Gascón, 2006, p. 55).

Instead, we want the student to understand how a mathematical idea arises inherently from the problem. It must be true because of logic and reality. Faced with the right problem, the student will conceive a useful mathematical notion, or generate an implicit model, as Brousseau sometimes calls it. "A piece of knowledge is the result of the student's adaptation to a situation, S, which 'justifies' this piece of knowledge by making it more or less effective … [for] the performance of tasks of different complexity" (Brousseau 1997, p. 98). A mathematical notion is identified by its usefulness solving problems, and it takes its meaning from that use and context

(Bosch & Gascón, 2006; McShane Warfield, 2014). This meaningfulness is the qualitative difference between learning a fact or procedure, which must be right because a teacher says it is, versus a concept whose justification is implicit in the logical structure of the situation. Brousseau says no less than the management of meaning "is the very object of the theory of situations" (1997, p. 266). Meaningful mathematical concepts understood in this way can be extended, deepened, and connected to other knowledge in further lessons.

Brousseau calls a lesson designed to stimulate a new mathematical notion a *situation of action*. *Action* because of the impetus required of the student. *Situation* because such meaning making is most likely to be realized by a student faced with not just a simply formulated problem, but a complex situation involving other students, the problem, previous mathematics and practices, and perhaps other outside factors. Crucially, the student must be convinced to search for the solution in the structure of the problem, in logic and reality, as opposed to expecting to memorize the right answers as taught by the teacher. To be discussed below are the conditions generating this student attitude, what Brousseau calls the devolution of adidactic situations, what modelers call authentic tasks, and teachers sometimes call relevance.

It may be helpful to give an example of a situation of action, that crucial initial problem where a desired mathematical concept germinates in the students' minds. The teaching of rational and decimal numbers was studied extensively by Brousseau (1997, 2013). A particular lesson he designed called "Sheets of Paper" introduces fractions as units of measure. The students already know the natural numbers, they can add and multiply, and they are familiar with using rulers. In the lesson they are challenged to differentiate six types of paper, ranging from very thin onionskin to thick card stock, as part of a game-like contest among small teams of students. To win, they must communicate through a curtain which stacks of paper are of which type, and they develop a written notation of thicknesses to do so. The sheets are too thin to directly gauge with a ruler, but stacks of 10 or 20 sheets are measurable. Number pairs representing the quantity of sheets in a stack and the stack's measured thickness become the notation. Of course two stacks of 10 sheets each are as thick as the two added together, and 30 sheets would be three times as thick. Hence these pairings of thickness and sheet counts--ratios--follow the addition and multiplication rules of regular numbers. But that will come later. The key according to the principles of the theory of situations is that ratios are born in the students' minds as a means to deal with the situation, not as dictated by the teacher (the adults refrain from any kind of conceptual assistance to the children). The future evolution of the students' understanding of ratios will always have at its root the meaning and significance born of their initial use.

Socially situated: the milieu or educational context. The student interaction built into the "Sheets of Paper" lesson example is a common feature in Brousseau's research. The idea that knowledge and learning is socially situated is not specific to Brousseau (Brown, Collins, & Duguid, 1989; Greeno, 1998; Schoenfeld, 1992), but it is a central feature of his theories, which he says, "model collective behaviors relative to the intended shared knowledge, and not just the knowledge and behavior of each student as an isolated subject" (2006, p. 7). Some lessons with student teams in competition require that mathematics be communicated in order to win. In others a convincing "proof" or "theorem" may have to be accepted by classmates in order to claim success. The students' explicit statement of mathematics serves two purposes. First, their understanding develops as the concept must be put in words, explained, defended. Second, the teacher (and researchers) can observe evidence of the student's knowledge. According to the theory of situations, designing the social interaction that generates mathematical knowledge is as important as selecting the mathematical problem itself. Brousseau uses the term *milieu* to encompass all of the elements the learner interacts with: the other students, classroom setting, teacher, and external features. In the “Sheets of Paper” problem, the students must communicate mathematics to other students through a curtain. They interact with their milieu, inventing and articulating mathematical concepts. A well-designed milieu distinguishes a routine mathematics problem from a productive lesson, a situation. “The term ‘situation’ designates the set of circumstances in which the student finds herself, the relationships that unify her with her milieu, the set of ‘givens’ that characterize an action or an evolution. ... [it] necessitates an adaptation, a response, by the student” (1997, p. 214).

The theory of situations seeks to identify and explain the optimal conditions for learning. At the core are the student, problem, and milieu. The teacher is an actor in the milieu, but he is also a designer, figuratively standing outside of the lesson with the goal to create a learning situation, “a system of conditions that make it probable that a group of students or an institution will produce a theorem or the solution of a problem.” “The notion of ‘situation’ includes, extends, enlarges, and diversifies the notion of ‘problem’” (2006, p. 2). The teacher has other, higher-level, perspectives, for example planning the unit, and the students, too, perceive themselves within a hierarchy of contexts. They are playing a game, but they are also aware they are in a mathematics class, they have course expectations, and so on. Thus the system of student, problem, and milieu can be thought of within nested contexts. The diagram in Figure 4 is representative of Brousseau’s depiction of the milieu and that of others that extended his work (Brousseau, 1997; Schoenfeld, 2012).

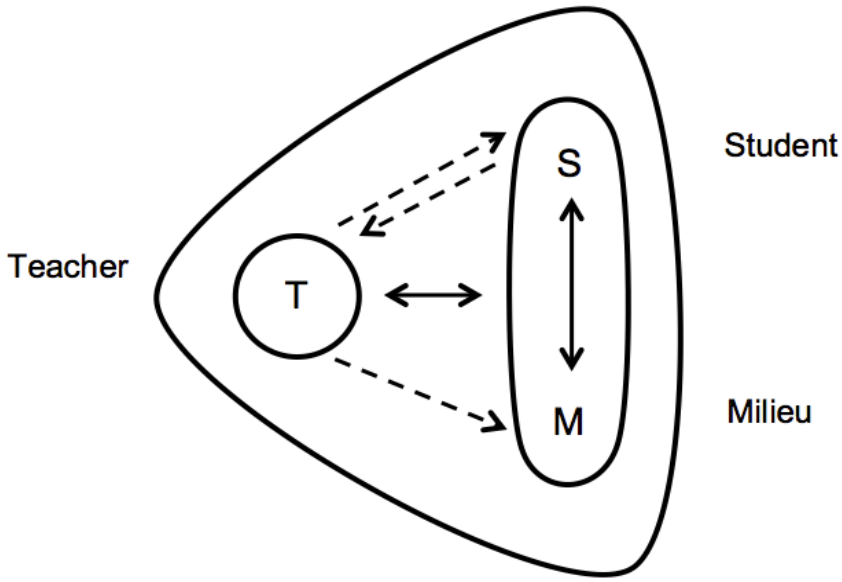

Figure 4: A representation of the interaction among the teacher, students, and mathematical milieu (Brousseau, 1997, p. 56; see also Schoenfeld, 2012, p. 590). Learning takes place as the student interacts with the milieu (vertical solid arrow) in a problem situation. The teacher's primary role is the organizer of the learning situation (solid horizontal arrow), but he also interacts with the student and participates in the milieu (dashed lines).

Didactical contract: modeling's unorthodox requirements of students. The teacher obviously has a special role in the milieu, both interacting with the students during the lesson and having designed it beforehand. Students and teacher develop a history together, forming what Brousseau with J. Centeno called a *didactic memory* (Brousseau, 1997), and they assume certain roles and practices. In fact, mutual expectations among students and teacher roles can be quite rigid, even if unstated. The importance of this relationship, what he called the *didactical contract*, is a central contribution of Brousseau. It represents the set of implicit, mutual responsibilities for teachers, students, and other stakeholders (e.g. parents) in the educational process. Brousseau showed that expectations can be contradictory and the didactical contract is paradoxically unworkable.

At the core of the paradox is the constructivist conviction, putting the student at the center of his or her own learning. The teacher wants to direct the student's thinking along an efficient path, but knows that deep understanding comes only through self-discovery. The student wants the teacher to explain the solution, but will take little meaning from it unless the student himself discovers it. Thus a successful classroom cannot function strictly under the teacher's authority, but that authority can not be fully suspended either. Traditional roles under the didactical contract are selectively violated and renegotiated. How this is accomplished is a key result of Brousseau's research. It is discussed below in the section Adidactic Situations and Devolution. It is also a rich area of overlap with modeling research and the reflections of this study's teachers teaching modeling lessons.

Classroom culture. Before discussing his theory in more detail, the following paragraphs offer examples of how Brousseau's main ideas of milieu and didactic contract continue to be useful. The theory of didactical situations remains a dominant theoretical framework in French mathematics education, particularly in terms of, as Schoenfeld puts it, "the core idea that classroom activities can be structured in ways that are inherently mathematical, and that reveal student conceptions to the teacher" (Schoenfeld, 2014, p. 66). Among applications in the English-speaking world, Herbst (2006) employed Brousseau's theory to study the teaching of geometric concepts in a high school class. The teacher's work is not just to provide suitable problems to students, but also to shape expectations, to cultivate a productive classroom culture, and to refine the didactical contract. That takes time. Reliable results depend on good problems along with the long-term development of a robust milieu, a classroom where students take responsibility, interact, and communicate mathematics. Herbst (2006) suggested that one reason mathematical modeling may be hard to teach is that students object to the requirement to explore and generate their own ideas. They may not see that as their role.

In a similar way, the subjects in this study discussed the unusual estimating and data gathering requirements of modeling, ideas that could be framed as a renegotiation of the didactic contract and expansion of the milieu. Indeed, Internet access in the classroom allows teachers to design lessons where students search for and make sense of actual data sets. Kent (2010) studied the impact of student computer use in a mathematics course using Brousseau's theories, particularly applying the concept of a milieu expanded by computer use. He opined that new technology held promise and recommended that it be studied as a adjunct to the traditional milieu's interpretation as a physical classroom setting. An online milieu may come to radically expand interaction and resources (Kent, 2010).

Framing the study of mathematics education around classroom culture focuses attention on what the Common Core calls mathematical practice standards, students' dispositions and mindset rather than the facts they know. Stating that we want students to model with mathematics implies that they look at the world a certain way, as ripe for quantification, logical analysis, and mathematical application. Brousseau's theory of situations entails creating such a classroom culture. Shaping children's dispositions and outlook is socialization, not skills building. Discussing guidance for the implementation of Common Core reforms, Schoenfeld concludes that Brousseau's framework takes in the right factors. "The answer, the author [Schoenfeld] believes, lies in much more of a cultural construal of the didactical triangle than is typically considered in the English-speaking world, and that has the breadth of perspective in the French scholars' work on didactique" (2012, p. 593).

Adidactical situations and devolution. Having explained the core of Brousseau's theory—constructivist learning situations, the classroom milieu, and a didactic contract between students and teacher—this section now discusses his major findings in terms of lesson sequencing, knowledge development, and obstacles and pitfalls.

Not all lessons entail student discovery. The initial student conception must be born in a special lesson, what Brousseau calls an adidactical situation, and that is only possible because the teacher turns over, or devolves, responsibility to the student to find the solution. Later lessons of different types solidify and standardize the students' concepts. The nature of students' understanding matures from vague intuition through increasing rigor conceptual understanding and procedural fluency. Certain obstacles are inevitable, and Brousseau categorizes them. Other common failings are byproducts of time pressure or weak teaching. Each of these findings is explained below, sometimes with mention of the fit to modeling research and always keeping in mind the goal of making sense of the study's teacher perspectives data.

As was explained above, mathematical understanding comes from solving a problem, not from listening to a fact or procedure as explained by the teacher. Learning is a modification of a student's knowledge through a process only the student can instigate. This requires a suspension of the usual authority of the teacher, a *devolution* as Brousseau calls it, such that the student believes that the solution is within the problem situation, rather than being a matter of the teacher's judgment of correctness. Then the student will make the effort to modify his own thinking, build an implicit mental model, and test it against the problem and constraints in the milieu. This special type of lesson, termed an *adidactical* situation, is successful depending of the attitude of the student, the decision to take responsibility for the solution (Brousseau, 1997; McShane Warfield, 2014). The devolution of an adidactical situation may include an element of theater or deception in its presentation by the teacher, but in Brousseau's work it also often follows from the social interplay among the students, the milieu, built into the problem. The students must compete or argue, and that gives them a stake, it creates a sense of realness in the situation.

The subjects of this study talked at length about the types of modeling problems that students found realistic, and therefore motivating and engaging. The nature of what modeling researchers call authentic problems and the importance of student beliefs and attitudes parallels Brousseau's ideas of adidactical situations and devolution. That literature is discussed below.

Institutionalization. Adidactical situations, where the student takes the initiative and constructs his own knowledge, are only the start. Brousseau also found that teachers insisted on a stage of review and reflection. In these lessons the teacher takes the lead, officially identifies the concepts the students have learned, puts them in relation to other concepts of mathematics, and notes their importance and status. The students have learned something, it is identified, and it will be useful in the future. The teacher steers the class toward conventional notation and terminology. This whole alignment of individually meaningful mathematics, collectively understood, is termed *institutionalization* (Brousseau, 1997, 2013; McShane Warfield, 2014). Brousseau rejects the possibility of an exclusively student-led curriculum, what he terms radical constructivism, because of the need for institutionalization. The curriculum must include a series of didactic situations (i.e. classical teaching lessons, as opposed to adidactical situations) in which the teacher resumes his natural position of authority (Brousseau et al., 2013).

Didactic transposition. In fact, institutionalization is just one example of the repeated interplay of classroom problems, individual concepts, and the overall canon of mathematics. The strategic practice of selecting concepts from the mathematics used in the adult, professional community, embedding it in school problem contexts, and then helping students abstract and generalize that knowledge was given the name *didactical transposition* by the group developing Brousseau's theory, most prominently Yves Chevallard beginning in 1980 (Bosch & Gascón, 2006; Brousseau, 1997). (As far as I know, Chevallard's 1985 book, *La transposition didactique*, is only available in French and translated to Spanish) An example of didactical transposition within modeling education is the modeling-as-vehicle versus modeling-as-content dichotomy: does modeling promote and motivate other mathematics, or is it a valuable competency in itself (Galbraith, 2007; Julie, 2002)? Society decides and debates priorities, approaches, content, and goals of the education system. How that happens and who takes what role is the subject of the theory of didactical transposition. The transformation of mathematical subject matter from discovery, to publication, to teaching is also the subject of the theory.

Brousseau uses the term *problematize* (also reproblematize and recontextualize) to describe the activity of selecting and developing problems to teach a particular topic. Good instructional materials begin with a mathematical concept to be learned as their goal, and then the authors "reorganize, reformulate, and 'reproblematize'" (Brousseau 1999, p. 44) that area of mathematics. He distinguishes that "didactic activity" from a teacher's "pedagogical activity" (1999, p. 77), that is, designing a lesson versus delivering it. "The teacher must 'remake' known mathematics, looking for the kinds of problem whose solution it facilitates, the sorts of question it causes to be asked, and how its efficiency and presentation can be improved. Recontextualizing mathematics in another way—in particular for students—is the teacher's essential activity. . .It constitutes a stage of the didactical transposition" (1997, p. 268; see also Bosch & Gascón, 2006; McShane Warfield, 2014). To solve the selected problem, the student constructs the desired mathematical concept. With use and discussion, the notion generalizes, abstracts, and integrates into the student's growing command of mathematical knowledge. The cycling between mathematics and problem contexts is reminiscent of the modeling cycle iteration of mathematics and real world. The processes have much in common.

Types of knowing: savoir and connaissance. In the decades of research at the Michelet school Brousseau (1997, 2013) investigated not just single lessons, but complete curriculum units. The researchers took a topic like decimals and fractions from initial conception through practice and mastery, thus tracking student knowledge over time. The theory of situations explains how learners' understanding develops and matures, and it includes a corresponding sequence of lesson types to optimally drive that development. The path to full development is not smooth. Brousseau discovered inevitable learning obstacles, and he described common but avoidable teaching errors. The sections below discuss these findings starting with the changing character of student knowledge, followed by the lesson sequence, and finally the obstacles and pitfalls.

Brousseau distinguishes two ways students may understand a mathematical concept, two ways of knowing. *Savoir* is the type of knowing when one knows a fact. It is concrete and communicable. *Connaissance* is more of an intuition, a sense of how ideas are related, when they are useful. Connaissances may not be conscious and will be difficult or impossible to communicate, but they are the conceptual latticework that gives the individual pieces of knowledge, the savoirs, significance and usability. "It [connaissance] is what gives meaning to the savoirs" (Brousseau et al., 2013, p. 131). Without meaning and appreciation of context, isolated facts and skills are difficult to remember or employ. The challenge then is to balance connaissance and savoir, or using the terminology of the Common Core (NGA & CCSSO, 2010), to develop both conceptual understanding and procedural fluency. The problem in schools, of course, is that facts and procedures are easier to assess and tend to dominate instruction. Brousseau repeatedly acknowledges modern pressures to "teach to the test," but he warns against of the impact: disconnected, fragmented knowledge that cannot be put to use in context (2013). (Of course, contextual judgment and interpretation are especially key for mathematical modeling in particular.) When educators attempt a shortcut by teaching facts and procedures directly, students learn to produce solutions based on what they think the teacher wants, based on cultural and social clues rather than the logic of the problem. Knowledge learned this way is temporary and unsustainable. He says, "We have seen the inadequacies of this 'mechanical' conception and the necessity of placing this use under the control of an understanding and even, as early as possible, of rational knowledge. Only the resolution of certain problem-situations can give this clear conception, this understanding and this knowledge" (Brousseau 1997, pp. 223-224).

In fact connaissance and savior are two ends of a continuum, more than that, a progression, from hunch to mastery. Brousseau observed that mathematical concepts are originally conceived as vague notions specific to the context that triggered them, connaissances. With use knowledge becomes generalized, concrete, and definite—ideas that can be discussed and manipulated, savoirs. Mathematical facts and procedures developed this way carry with them the significance and meaning of the contexts in which they were first constructed by the learner. That meaningfulness can be woven into a rich conceptual understanding of the world as seen through logic and mathematics. The details of a program of ordered progression from discovery through development, from connaissance to savoir, are the subject of Brousseau's research. "What are the reciprocal roles of habit and understanding in the acquisition and functioning of a notion? … What situations and what behavior correspond to a suitable appropriation of a concept? What are the errors that appear, and their meaning?" (1997, p. 224). The result is the theory of situations.

Curriculum as a succession of situations of different types. At the Michelet school several curricula units were developed and studied, optimal sequences of different types of lesson situations to lead students through this path from connaissance to savior. Brousseau's curriculum stimulates students to formulate new mathematics, at first just a sense, an intuition, but step-by-step reinforced in different contexts—especially through communication among students in the classroom milieu. Further lessons call for the informal statement of the mathematical notion to become more rigorous, then be proven or logically supported. Finally the students can discuss and employ the concept at will, and through the process of institutionalization they use standard notation and learn of its significance in the overall edifice of mathematics. Within a curriculum unit, different concepts can be in different stages of development at the same time, accumulating and maturing in parallel (Brousseau et al., 2013).

His curricula are composed of a sequence of situations with the following names: situations of *action*, *formulation*, *validation*, and *institutionalization*. Eliciting a new mathematical idea is the first, most difficult lesson. Such situations call for action, invention. "They are many-faceted adventures that pull together a whole conglomeration of connaissances that will be provoked, activated, invented, used, modified, and verified, around a project of a mathematical nature dealing with an essential mathematical notion" (2013, p. 131). The concept is justified and demanded by the action that it enables of the student to solve the problem. Situations of formulation follow the initial lesson: the initial vague mathematical intuition in the student's mind is reinforced and solidified through communication among the students and through reuse in slightly different contexts. Finally through situations of validation and situations of institutionalization the concept can be discussed and debated as a mathematical object and understood within the official mathematical hierarchy with standard notation and terms (Brousseau, 1997).

Brousseau's methods rely heavily on student interaction. Concepts first occur "as an implicit solution to an 'action problem' so as to follow up with communication problems which would explicitly need a designational system" (1997, p. 257). His methods are consistent with research on mathematical discourse in the classroom and professional community, that mathematics is a social practice: students articulate their ideas (formulation), they convince classmates of their truth (validation), and they adopt standard terminology and notation (institutionalization, McShane Warfield, 2014).

Brousseau remarks, interestingly, that it is the failings of the students' approach that push forward their understanding. Errors, contradictions, and limitations cause revisions to implicit models and deepening of understanding. Generalization and abstraction resolve conflicting examples. Thus too careful or too neat a development can be a disadvantage. Better the occasional crisis, debate, or controversy (perhaps intentionally incited) for conceptual development, and motivation and interest as well (Brousseau, 1997). Here again, the social setting, the milieu, is the forum triggering these benefits.

Obstacles. The theory of situations reveals the source of several types of difficulties in education. Some are common but avoidable errors, like yielding to pressures to teach to the test. Several types of problems are inherent to mathematics or the way it is taught. The following paragraphs discuss these obstacles, as Brousseau calls them.

A coherent mathematics curriculum across the grades is challenging to develop not just because it is a large, complex undertaking, but because the progression is not a cumulative process. At many points previous concepts must be revised, often radically. Brousseau gives the example of transitioning from arithmetic to algebra. In arithmetic, students learn that “3+4=7” means, “if you add three and four you get seven.” Later in algebra when we say, “3+x=7 implies x=4,” the equal sign’s meaning, procedural role, and conceptual basis are completely different. “Everything has changed with respect to these familiar symbols. ... The pupil must therefore not only learn new knowledge, but also re-learn and organize old knowledge and forget—at least un-learn—part of it” (1997, p. 254). Unfortunately, in school this revision process is not usually explicitly discussed. Rather, the new version of correctness is taught and the student must identify a change and reconcile it with prior understanding. Brousseau says critically, “Curricula developed today provide nothing but the juxtaposition of blocks of learning activity” (1997, p. 254). Instead, educators should recognize these learning obstacles and address them methodically. “The identification and characterization of an obstacle are essential to the analysis and construction of didactical situations” (1997, p. 77).

Brousseau found that obstacles arose for a variety of reasons, which he categorized into three types. Some are inevitable because of the structure of mathematics itself, for example the limited meaning of the equals sign learned in grade school that later can become an obstacle to more advanced work, discussed above. These he called *epistemological obstacles*. Other limitations are natural results of students’ age or development, “obstacles of onogenic origin.” Other problems crop up because of notation or how concepts are generally taught, “obstacles of didactical origin” (1997, p. 86). McShane Warfield states that our high stakes testing in high school of very abstract notions only vaguely grasped is an example of such an obstacle of didactical origin. The mismatch between students’ abilities and testing demands stresses and undermines the educational process (McShane Warfield, 2014). To give an example for mathematical modeling, if we teach students to identify all of the given quantities in a word problem and then relate them with an equation, they learn a limited method that will be an obstacle in modeling when many quantities may not be given.

Didactical fallacies, paradoxes. In his years of research, Brousseau identified a number of puzzling and interesting phenomena. He often gave colorful names and descriptions to these paradoxes, teaching errors, and confounding practical tendencies. Many are applicable to mathematical modeling, and they are discussed in the following paragraphs.

There is a contradiction between constructivist learning and the didactical contract (the standard roles expected of teachers and students), which Brousseau calls the *paradox of the devolution of situations*. The student must accept the teacher's authority over the work to be done, the terms of the problem situation for example, but the student must then work independently, searching the logic of the problem to construct his own mathematical knowledge. In a sense therefore, successful devolution requires a mutual acceptance of the teacher's authority and a suspension of it. In practice Brousseau suggests that the paradox is often overcome through theatrical presentation, lighthearted deception, or playful game situations (Brousseau, 1997; McShane Warfield, 2014). In those cases the students and teacher accept that the situation may have contrived elements; it is after all a lesson for school, but the class takes it as given and throw themselves into its solution with commitment. Success follows in part from a productive classroom culture, something that is created over time.

On the other hand, a history of rote learning undermines the practice of Brousseau's techniques. If students are taught mathematical procedures by repeating the teacher's examples, then they will not select them based on logic, but instead by recognizing the contextual clues in the presentation of the problems. Brousseau claims that it is very hard for a student to apprehend the mathematical significance of a concept later, or to reacquire productive beliefs and attitudes about one's own responsibilities and efficacy. "Memorization of formal knowledge, largely meaningless, can be costly…. The representation which the student makes of the mathematical knowledge is profoundly perturbed as a consequence. The more the student has been drilled in formal exercises, the more it is difficult for her, later, to restore a fruitful functioning of concepts so acquired" (1997, p. 43). The student cannot employ mathematics based on logic and insight since she has learned to associate concepts with artificial school-based hints in the question's presentation. In Brousseau's words, the student selects procedures by "decoding the didactical convention" (1997, p. 35). The subjects of this study cited this challenge to motivating students. For true modeling, the real-world structure of the problem determines the mathematics, not school context, but students accustomed to adhering to formulaic methods will object to less deterministic problems. One solution is creativity or drama when posing the problem.

A skillful teacher sometimes becomes an actor at the front of the classroom to pose an engaging problem. The conflicting need for flexibility while following a plan leads to another of Brousseau's paradoxes. On the one hand the teacher must follow a script, follow the plot toward the educational goal of the lesson. On the other hand, students' ideas and responses must drive the pace and direction of the lesson. It is their initiative that is needed for learning after all. This tradeoff Brousseau calls the *paradox of the actor*. The teacher must invest himself in the situation and desire its solution, but also cannot be the primary participant if he is to devolve the responsibility to the students.

Likewise, the teacher's personal engagement with the problem itself is a factor in successful devolution, successful acting. He will be better able to engage and motivate the student's interest if he himself is engaged in the problem, if it is novel and unsolved for him as well, but of course he must know the underlying mathematics well enough to plan and present the lesson. Thus, good problems and teaching require a balance of control and uncertainty in the teacher's knowledge of the problem's solution. Brousseau's long history of experimentation allowed him to discover that sometimes the lessons became less effective over the years, a phenomenon he called the *obsolescence of didactical situations*. The teachers' familiarity with the problems, he speculated, caused a subtle shift in their delivery, inadvertently giving the students clues about the desired results (Brousseau, 1997).

Brousseau examined and rejected shortcuts to the methods of the theory of situations. For example, in the 1960s, reforms led by mathematicians suggested teaching abstract mathematical concepts as the starting place, which children could learn first and from there practice applications. In contrast with Brousseau's theory of learning, where students' initial conceptions were tightly associated with the particulars of the situation requiring their invention, the direct teaching of general mathematical structures would, in theory, be more efficient and avoid the need to correct limited initial concepts, in other words eliminate epistemological obstacles. Unfortunately, the claim that students really grasped the generalized mathematical structures attributed to them was unsubstantiated by experimental evidence, and Brousseau criticized the surface appearances justifying the structuralist's claims. Brousseau (1997) cites an example lesson of a young child flipping and rotating a doll through various symmetries under the direction of the teacher. Ludicrously, the teacher then declares that the student has represented the Klein group! (p. 139) Obviously the third grader is far from really understanding abstract algebra. Brousseau asserts that this temptation to expediently tell the student the answer and misattribute conceptual understanding is all too common in schools. He poked fun at the phenomenon with stories like these grade school students mastering abstract algebra and with fanciful names alluding to French classics, the *Topaze effect* and the *Jourdain effect*. In each case, the teacher's leading questions and profuse praise of the student's correct answers make a mockery of the struggle required for true competence. Independent understanding is completely sacrificed to the desired progress of the lesson, and the whole exercise becomes a farce (Brousseau, 1997).

Like leading questions, the repeated use of similar examples until the student recognizes the pattern is not a substitute for carefully constructed situations where logic is the basis of the student's understanding. This error is called the *abusive use of analogy*. Brousseau says, "This abuse of analogy led the student to look for resemblances corresponding to the teacher's intentions and to focus upon irrelevant variables instead of understanding the internal needs of the situation. Thus, she solved her problems more by transfer of algorithms than by understanding the meaning" (1997, p. 270). Some students will identify the common mathematical logic in the examples, but many will not. Teaching abstract and general pieces of knowledge directly will not build a rich, interconnected understanding for most students.

Brousseau sees no shortcut around the inefficient truth that like the historic development of mathematical concepts, which were tightly linked to contemporary conditions, students' initial concepts are particular to the problems and conditions that generate them. A subsequent sequence of situations reinforces, generalizes, and scrubs the original notions until they are finally under the students' command in generalized, abstract form. At that point they can be aligned with standard notation and placed in relation to the students' other mathematical knowledge. This process is orchestrated by the teacher, necessarily so. To assume that institutionalization can occur without the intervention of the teacher is an error Brousseau called the Diénès effect (after Zoltan Diénès, another educational theorist).

Research methods and didactical engineering. This literature review has now covered a description of Brousseau's theoretical and experimental results: the theory of situations, which is essentially constructivist, situated, and cultural. Two topics remain. The first is based on his research methods—how they apply to this study and might inspire the development of curricula for modeling. The second is Brousseau's view on the teaching of mathematical modeling specifically. In fact, we are fortunate that he outlined a unit to teach estimation techniques for use with modeling.

Three of this study's research questions involve the design of curricula models (the modeling cycle), instructional materials, and collaboration. Brousseau (1997) would label this as falling within didactical engineering, the practice of practical design and implementation of better instruction. It is the "study of the conditions of production and transmission or re-production of mathematical thought" (2006, p. 9). His decades of lesson design and implementation at the Michelet school serve as a pattern to guide modeling curriculum and instructional materials design, in particular his careful selection of problems based on the mathematics to be learned.

When we write modeling lessons the content standard is the starting place and the real world problem that employs it follows. "A 'mathematical structure' takes its meaning from the use which is made of it, from its function, from the constitution of other and above all from the problems whose solution it permits" (1997, p. 144). Juxtaposing his methods with those of structuralists (e.g. Diénès), who would order instruction to follow mathematical logic, Brousseau explains that in the logical sequence of design [didactical engineering], one must "question mathematics in order to search beyond the structures for the concepts and beyond the concepts possibly for the conceptions which could be built up by a subject [student] in particular historical or didactical situations”(1997, p. 144). The best modeling problems are not just an application of a concept or procedure, therefore, but they are a situation that gives meaning to the mathematics, one that evokes a need felt by the student. Programmatically Brousseau asks lesson authors, “How can we elaborate situations which really make a notion function?” (1997, p. 147)

In a retrospective keynote address in 1999, Brousseau lamented that while scientific knowledge of the didactics of mathematics has advanced, that has not diffused into practice (1999). He is at times pessimistic about the distribution of new, better lessons to teachers for both practical and social reasons. It takes many pages to describe a situation in detail (for example, 30 pages, he says), which few teachers will have the patience to read. Secondly there is not a precise vocabulary that is known to laymen teachers. Finally, teachers are the acknowledged local experts with very diverse practices and little centralizing tendency that would lead to standardization (Brousseau, 1997).

Methods that integrate the research and teaching functions to varying degrees may be one answer. Action research combines teaching and development, with the explicit goal of improvement, taking action (Brousseau, 1999). The interviews employed in this research follow the spirit of this commingling of research and teaching roles, though without action in the classroom, what Brousseau called “participatory observation” (1999, p. 42). The researcher himself is a teacher, and both teachers are attempting to better understand mathematics education and improve their practice of it, through reflection on their experience.

Brousseau cautions that a critical step in educational improvement is validation: policy must be implemented and then tested empirically. In particular, reform has to be tested against actual impact on practice, that is, the form it takes among practitioners. “How does it manifest itself within the community? How is it discussed and transformed…?” he asks (1999, p. 41).

Hence teacher perspectives, like the ones gathered for this study, are a key step in the validation of modeling research and the rollout of modeling policy.

Regarding the role of teacher input, it is interesting to hear Brousseau retell his discovery of institutionalization, a key stage of instruction. The experimental method employed at the Michelet school was a collaborative one: researchers drafted the lesson situations and teachers employed them with students so they could be observed. At one point, however, the teachers insisted something was missing in the curriculum. They wanted to design and include some lessons of their own, for purposes not yet recognized in the theory of situations. One can imagine the dynamic between the teachers and the researchers. Brousseau explains, “It took us some time to realize that they really needed to do some things, for reasons that had to be understood. ... It wasn't a question of judging them or their methods, but of understanding what they legitimately needed to do and why they needed a degree of opacity in order to do it, faced with researchers” (1997, p. 236). In that spirit, this study of the implementation of the Common Core modeling standard looks at teachers’ perspectives for what may be missing, what is of practical importance, and to understand how modeling is taught and learned.

Brousseau’s comments on real world problems, modeling. Brousseau’s comments on mathematical modeling highlight problems with real world background knowledge, estimation as a skill, and validation. He views the call for more real life problems in mathematics education an important “cultural movement” (this is during a talk in 1988, Brousseau, 1997, p. 241) that would require study through the application of the methods of the theory of didactical situations. He says, “Pedagogues extol research of situations which allow the child to be put into contact with real problems. But the more the situations of actions realizes this contact with the reality, the more complex are problems concerning the status of the knowledge” (1997, p. 239).

Brousseau notes the modelers' arguments for these problems: "'hands-on activities make for better understanding and better learning; 'reality avoids errors of understanding;' 'usefulness, the concrete, motivate the student'" (pp. 240-241). But versus an idealized problem situation, is it worth it? "'Reality' is very much more difficult to 'understand' than a theory," he says (1997, p. 241). Real measurements and other actual conditions generally never exactly match the mathematical ideal, so how do we determine correctness? Has the student taken into consideration the proper real world criteria in the validation of his solution?

At the core of the problem with modeling is the incorrect teaching of estimation in elementary school, or rather than incorrect teaching, estimation is an example of an epistemological obstacle. Arithmetic is taught with frequent examples of measurement, but the imprecision of the real world is ignored in favor of the exactness of formal mathematics. "If a student estimates that 3+4=6, the teacher doesn't tell her that she is not far wrong" (1997, p. 239). Calculations are exact, and in grade school so are the quantities of a fictional reality. In later work this simplification must be replaced with a proper understanding of measurement errors, uncertainty, intervals, precision, and so on.

This mathematical skill with estimation is important to modeling. As a proof-of-concept Brousseau outlines a lesson (situation) grappling specifically with the concepts of approximation students must master to successfully solve real world problems. The class is hypothetical; Brousseau did not experimentally develop a modeling curriculum, but he thinks it would be possible using the techniques of the theory of situations: "arrange a process of activities, of communication of results, of exchanges of guarantees, of reflections and of discussion" (1997, p. 243). The comments of several research subjects regarding estimation highlight that this area remains a gap in current day instruction.

The Alignment of Brousseau's Theory and Teaching Mathematical Modeling

The next section of the literature review aligns Brousseau's theory with major ideas in the modeling literature. The brief histories of the two lines of research show that they developed in parallel over the last 50 years. Brousseau's was both more focused geographically, with its nucleus in the research team and experiments done at the Michelet school, and broader in scope, applying to mathematics education generally. In modeling, the research community enacted what Brousseau's school termed didactical transposition: practitioners, educators, researchers, and policymakers decided what mathematics should be taught and how it should be shaped for the classroom. The culmination of that effort for the United States is the Common Core modeling standard and modeling cycle (NGA & CCSSO, 2010). The promise of a synthesis of the two bodies of work is that if key areas of teaching modeling can be mapped appropriately onto the theory of situations, we will benefit from the coherence of Brousseau's theory as well as all of the results of his experimental findings and their implications for implementation. This study's first research question concerns the feasibility of that goal: How well does Brousseau's theory of didactical situations align with our current understanding of the modeling cycle?

The following pages compare matching elements of modeling and the theory of situations and provide a short exploration of the latter's greater detail, integration, and practical findings, highlighting the potential to view modeling through the vantage of Brousseau's work. This summary is intended to serve as a bridge between the two areas of research.

Concepts versus skills, Brousseau's ways of knowing. Brousseau distinguished *connaissance* from *savoir*, two types of knowledge. The modeling literature contains a similar distinction between conceptual understanding and procedural fluency (NCTM, 2000; NGA & CCSSO, 2010). For example, early in the modern calls for modeling in education, Pollak (1969) explained the benefits of teaching mathematics as a process rather than emphasizing getting the right "answer." "Nevertheless, it is terribly important for students to have practice in seeing situations in which mathematics might be helpful, and in trying their hand at formulating useful problems. In fact, one of the most valuable lessons which comes from trying real applications of mathematics is that finding a problem that is 'right' for a particular fuzzy situation is itself a real mathematical achievement. This is important in the classroom not only because it is the honest truth, but also because it helps to de-emphasize the 'answer' as the sole goal of mathematics, and helps to shift emphasis toward mathematical structure and process" (p. 400). On the other hand, modeling can be taught as a skill, breaking it into steps and applying it to standard situations of increasing difficulty. Contrasting the alternatives, Burkhardt summarized the skills approach as the case where students are "learning about standard models of a wide range of phenomena, rather than active modelling by the students of new situations. Each of these activities is important; the second is more challenging, both for teachers and for students" (1989, p. 159). Brousseau said that the distinction between two types of knowing was a simplification of what, in fact, was a continuum, and his theory explained how instruction moved students from *connaissance* to *savoir*. The theory of situations offers a roadmap for curricula that develop procedural fluency from initial conceptual understanding. As a detailed theory connecting the two types of knowing from a pedagogical point of view, it may be a helpful framework for modeling curriculum authors and teachers.

Real-life problems versus situations. The central role for problems in Brousseau's methods, or situations as he calls them, has several parallels in the modeling literature. Situations are problems to be solved, but they also encompass the curricular and social context, which Brousseau showed are critical to stimulate mathematical understanding. For students, the mathematics they learn takes meaning from the context in which it is employed. Similarly in the modeling literature, Niss, Blum, and Galbraith go so far as to claim that, at least for younger students, a real world application is almost a requirement for conceptual learning: “It is difficult to motivate or learn mathematical concepts, methods, techniques, terminology, and results and to engage in mathematical activity, unless clear reference is being established to the use and relevance of mathematics to extra-mathematical contexts and situations, which are often also responsible for creating meaning and sense-making with regard to the mathematical entities at issue” (2007, p. 5).

Modeling addresses problems of a particular kind, those from the real world, and such application requires actions that students do not generally employ for school problems: making assumptions, interpreting solutions in the real world context, and judging validity. Teaching modeling is therefore more difficult than teaching routine exercises. “Handling non-routine problems in the classroom presents teachers with substantial challenges, both mathematical and pedagogical, that are not met in a traditional classroom” (Burkhardt, 2014, p. 9). Brousseau employed highly-structured lesson descriptions in his research. Similarly, in his lesson development Burkhardt found that narrowly specifying the teacher’s actions led to lower pedagogical demands for teachers, but could still provide active modeling opportunities for students. “An essential element of success was the discovery that materials can be quite directive as to the pattern of activity in the classroom, while leaving the actual solution of the problem as an essentially open task” (1989, p. 160).

Estimating input values is a frequent modeling requirement sometimes left out of traditional school mathematics. In his criticism of the traditional treatment given modeling in textbooks, Pollak highlighted the special role of estimation. “One of the first points about which to be honest with the student is the precision of the data. Rough and approximate calculation is not only excellent mathematical practice, but may, in fact, be the only justifiable response to the approximations made in obtaining the mathematical model of reality which the problem represents" (1969, p. 395). Students will be frustrated or protest when modeling violates tacit assumptions of what mathematics class should require. Brousseau's study of the didactic contract, the importance of student and teacher role expectations and the dynamic when those expectations conflict with what is required for a lesson, may help clarify how to manage adding to the curriculum the skills necessary for modeling.

Socially situated learning and Brousseau's milieu. Students make meaning of mathematics together with other students. To cultivate a productive classroom one must take into account social aspects of learning, the milieu. What differentiates Brousseau is his extensive use of social factors for specific functions in lesson situations. Brousseau's game-like situations require students to work in teams or, in some cases, to take competing sides. Mathematical argument sharpens students' thinking, especially when convincing one's peers determines success in a contest. Real life problems may offer the stimulation of social interaction in a way similar to those of Brousseau's lessons. The modeling research community has come to view mathematics instruction in a social context. As Schoenfeld recounts, "Finally, the tail end of the 1980s saw a potential unification of aspects of what might be called the cognitive and social perspectives on human behavior, in the theme of enculturation. The minimalist notion of this perspective is that learning is a social act, taking place in a social context; that one must consider learning environments as cultural contexts and learning as a cultural act" (1992, p. 347).

Brousseau's methods involve more than engaging group work. Peer communication, argument, and evaluation serve specific functions in his lessons to elicit and refine student knowledge. In our current modeling research, we also frame students as participants in a classroom community. “There is an emerging epistemological argument suggesting that mathematical collaboration and communication have a much more important role than [previously appreciated]. According to that argument, *membership in a community of mathematical practice is part of what constitutes mathematical thinking and knowing*” (1992, p. 344). This parallel suggests that lesson authors use the milieu explicitly as a design element. For example, students might be required to validate a model by proving to peers, rather than to the teacher, that a solution is reasonable.

The authenticity of modeling problems and adidactic situations. Selecting good problems—that engage and promote learning—is a focus of any mathematics education program. Modeling researchers have used several terms to characterize desirable problems. “Authentic” problems (Galbraith, 2007; Niss et al., 2007) have real importance in the outside world. “Honest” problems (Pollak, 1969, p. 401) are those a student might reasonably be expected understand and solve. “Usable” problems (Julie, 2002, p. 4) refer to a student’s sense that a problem might actually pertain to his or her life. Brousseau’s distinction between adidactical situations (problems) and regular teaching lessons overlaps with the modelers’ criteria of authenticity, honesty, and usability, but Brousseau focused on a critical pedagogical feature, that the student seeks out the mathematical logic of the problem, as opposed to teacher- or school-sourced direction. This refinement led Brousseau to an extensive exploration of how teachers can successfully shape students’ viewpoint in this way, the phenomenon or practice of devolution. Thus the theory of didactical situations explains not just what is a good problem, but how the teacher’s delivery causes it to be good in the first place, by influencing student beliefs.

Student initiative and responsibility in terms of the didactic contract. Students reinforce a "productive attitude" when they successfully model problems they think are important. As knowledgeable actors in the real world themselves, their judgment of the validity of their solution is somewhat independent of the teacher's authority. Such acceptance of responsibility is central to the conditions Brousseau claimed assign mathematics meaning, and therefore to students' conceptual understanding of mathematics. Modeling researchers also value an approach where students act independently and take responsibility, although they use different terms, such as the discovery method or an investigative approach. For example, according to Pollak, "It is important to realize however, that formulating the 'right' problem in a situation outside of mathematics is a creative activity much like discovering mathematics itself. Thus our continuing efforts to bring the discovery method to the classroom naturally go hand-in-hand with attempts to bring genuine applications to the classroom" (1969, p. 404). Burkhardt warned that these methods are more difficult. "The investigative approach that is essential for the learning of active modelling places far greater demands on the teacher" (1989, p. 159). Brousseau's explanation of the didactic contract suggests that student attitudes must be negotiated, or shaped over time, and not expected to develop automatically.

Competencies, curricula sequencing, and institutionalization. Brousseau contested the radical constructivist claim that education requires nothing more than that students grapple with rich situations in a productive milieu. Yes, that is the way mathematical notions originate and take meaning, but they must then be practiced and shaped to conventions, a process he called institutionalization (1997). If we want students to be able to solve real life problems, if we want to teach modeling as content, we need to start with what the literature calls modeling competencies, tasks within the modeling cycle, and then go through a rigorous development of pedagogical detail and instructional materials (Niss et al., 2007). Burkhardt and Schoenfeld call such a disciplined design of materials and processes “engineering research,” in the sense that it is “directly concerned with practical impact” (2003, p. 5). Their prescription for mathematics education is a higher status for engineering research and more of it: “More engineering research. Researchers in education can contribute a great deal to the engineering research approach— the imaginative design and systematic research-based development of educational materials and their implementation” (2003, p. 9). Brousseau’s theory of situations can be a design framework. It specifies a sequence of lesson types beginning with eliciting a notion and ending with institutionalization. The Brousseau’s work offers both a detailed guide and a striking illustration of the shortfall in our modeling curriculum.

Chapter III: Methodology

The purpose of this research is to study the implementation of the Common Core modeling standard by applying the theories of Guy Brousseau when exploring teachers' beliefs about, and instructional practices related to, mathematical modeling. The adoption of the Common Core heightens the priority of instructing students to apply mathematics to real world problems. It is the intention of the study that the perspectives of a sample of teachers with experience teaching modeling will help textbook authors, teacher educators, and school administrators develop support and resources that teachers will value and use. Brousseau's pedagogical theories provide a theoretical framework to better understand the study participants' responses, and the extension of his work to a modeling context adds to the body of theory supporting modeling instruction.

The chapter is organized into seven major sections. The first section justifies the qualitative research approach used in the study. The next three sections describe the participants of the study, give an overview of the information needed to address the research questions, and explain the research design. How the data were gathered in interviews and how they were analyzed are described next, followed by a section covering ethical considerations, issues of trustworthiness, and limitations.

Justification of Qualitative Research Approach

Teachers have a central role in this study because they are the ones who must implement the Common Core policy. They select the instructional materials that publishers provide, and they decide whether and how to collaborate with each other. Teachers first of all are decision makers. Secondly, they have extensive knowledge of what actually works with students and is or is not feasible in the current educational system. Therefore, understanding their perspectives offers the means to answer the research questions and achieve the study's purpose.

The complex issues addressed in the study—the modeling cycle as a pedagogical framework, the suitability of instructional materials, collaboration—interrelate with many other factors; teachers' responses to the research questions will be context-specific. Qualitative methods are best suited to understand people's (teachers) perspectives, social phenomena which are not quantifiable or easily broken down into a reasonable number of variables (Creswell, 2013; Merriam, 1998; Yin, 2009). A multicase study design allowing several perspectives to be collected and compared was employed with the intent of gathering robust findings that might be representative across various school settings, professional backgrounds, and experiences. The method follows what Yin calls a "replication logic" (2009, p. 54). With a small number of subjects the results will not necessarily be generalizable, rather the goal is "transferability" of the findings, that they "apply or be useful in other similar contexts" (Bloomberg & Volpe, 2012, p. 36). Grounded theory methods were used in the analysis because they are appropriate for synthesizing common conceptual categories among various perspectives into theory with wide applicability (Charmaz, 2014; Glaser & Strauss, 2009). A complete grounded theory study would require resources beyond the scope of this study; thus this is essentially an exploratory, multicase interview study with elements of grounded theory.

Participants in the Study

Six high school mathematics teachers participated in the study. A sample size of six subjects is large enough to gather a variety of viewpoints while being a practical size to analyze carefully. There is no strict consensus as to the optimal number of subjects in a multicase study. Creswell recommends not to "include more than 4 to 5" (2013, p. 157), Yin argues for "6 to 10"

(2009, p. 54), and Merriam admits, “There is no answer” (1998, p. 64), saying it depends on the study’s research questions, data, and the resources available.

Purposeful sampling was used for this study, with convenience being a significant factor (Creswell, 2013; Merriam, 1998; Yin, 2009). I selected teachers that I knew personally and that I thought would be motivated to share candid views on the research issues. This would allow me to develop a rapport in our interviews quickly and collect thoughtful responses. Another criterion the teachers shared is that they all had professional experience or a relatively sophisticated knowledge of modeling, and they were practiced in teaching it in their high school classrooms. For example, one worked as an economic modeler before becoming a teacher, one is trained as a PhD-level engineer, and three are computer programmers. The intent was that their technical backgrounds would enable them to speak more knowledgeably about the subject matter of the study than would other teachers lacking such experience.

On the other hand, the participants teach in a variety of settings and schools, giving a broad set of viewpoints useful for cross-case comparison and more likely to lead to results that will be widely applicable. The participants’ backgrounds are summarized in Table 1. Their teaching experience ranges from four years to thirty years. They teach in locations across the United States, and most, but not all, teach in urban areas. Their school types and associated student bodies vary across public, charter, private, and selective schools.

Overview of Information Needed for the Study

Three general types of information were needed for the study. The perspectives of the participants, as well as information about their background to put their views in context, were the primary data of the research. The study’s multicase design is suited to gather this type of data.

Table 1

Background of the Participants in the Study

Subject	Years teaching	Courses taught	School	Setting	Location
Alice	5	Algebra, Geometry, Algebra II/ Trigonometry, Precalculus	Charter, urban	Urban	Boston
Belle	7	8th grade Mathematics, Algebra, Geometry, Algebra II/ Trigonometry, AP Calculus	Public	Small town	North Carolina
Carol	30	6th-12th grade mathematics of all types, Computer programming	Private Waldorf	Sub-urban	Northern California
Denise	4	Geometry, Precalculus, AP Computer programming	Selective public	Urban	New York City
Ellie	4	7th grade Mathematics, Algebra, Geometry, Algebra II/ Trigonometry, Precalculus	Public, International Baccalaureate	Urban	New York City
Fred	4	Algebra, Geometry, Precalculus, Computer programming	Public	Urban	New York City

Theoretical information to help understand both the subjects' perspectives and the context of teaching mathematical modeling is the second type of information. This took the form of a conceptual framework integrating Brousseau's theories with the literature on modeling instruction (including the Common Core). The construction of a conceptual framework to guide the data gathering and analysis is also characteristic of qualitative research (Maxwell, 2012; Merriam, 1998). That the conceptual framework continues to develop concurrently with the collection and analysis of data is a feature, especially, of the grounded theory method (Charmaz, 2014; Glaser, 1978).

Modeling lesson materials were a third type of information needed to support the data gathering. Early in this research I became involved with what developed into a three-volume set of modeling lesson textbooks. I co-authored an assessment unit for the Mathematical Modeling Handbook II (Vialva & Huson, 2013), and wrote four "lesson paradigms," or teaching chapters, for the third volume ("Handbook III," Huson, 2015). My preface in Handbook III discussed how Brousseau's theories could be applied to modeling instruction. It was the first version of what became the conceptual framework for this study. Two of the lessons treated in Handbook III, "Narrow Corridor" (Tan, 2012) and "Sunrise Sunset" (DePeau, 2012) were used in the research interviews to focus and stimulate participant responses. A third modeling problem, "Bale of Straw" (Kaiser, Lederich, & Rau, 2010, see Appendix C), from a modeling course I attended under Borromeo Ferri with Blum was also used in the interviews. In addition, study participants were invited to discuss with me their own materials during the interviews. Three subjects then provided me with modeling lessons or student solutions after our meetings. Thus the lesson materials played a role in the data gathering and theory development, and they were, in a sense, a result of those processes as well.

The information needed to answer each research question varies, and, therefore, for completeness, those requirements are described for each research question below.

1. How well does Brousseau's theory of didactical situations align with our current understanding of the modeling cycle

This is answered primarily by the Literature Review chapter relating Brousseau's theories and the research literature on modeling. The fit of Brousseau's theories to modeling instruction was also shown by the preface and modeling chapters published in the Mathematical Modeling Handbook III (Huson, 2015). Finally, the participants' comments often affirmed or modified ideas from Brousseau's theory in the context of modeling instruction (although they were not aware of his work nor did they use his terminology).

2. To what extent do teachers consider the modeling cycle as an important framework to structure their instruction? How, if at all, do they report it influences the way they prepare to teach mathematical modeling and assess students?

The subjects' reports of their opinions and experiences were analyzed to answer this question. The Common Core modeling standard (NGA & CCSSO, 2010) was adapted for use in the interviews.

3. What are teachers' perceptions of the appropriateness, ease, and usefulness of mathematical modeling lessons? What additional resources, if any, do teachers report as necessary for the teaching of mathematical modeling?

Findings for this question are based on participant responses with respect to the three modeling lessons presented (DePeau, 2012; Kaiser et al., 2010; Tan, 2012), their own materials, and other resources they cited as necessary or useful.

4. In what collaborative activities do teachers engage while planning, implementing, and evaluating mathematical modeling lessons? What additional forms of collaboration, if any, do teachers report they would participate in if they were available?

Teacher responses were used to answer this research question regarding collaboration.

Research Design

This section describes the study's research design. It begins with a summary of the multicase study methods employed and the elements from grounded theory. Second, a chronology of the initial modeling research, proposal development, and approvals provides background to the early stages of the study. Third is a discussion of the conceptual framework and related publication, Mathematical Modeling Handbook III (Huson, 2015), which was used to develop the interview protocol and support the data analysis. The conceptual framework also guided the selection of lesson materials that were an element in the interviews, which are discussed in the final part of this section. The interview methods to collect data and the data analysis are explained in detail in their own sections.

Interviews with elements of grounded theory. Multicase study methods guided the research design: open-ended interviews, cross-subject comparisons, triangulating subject responses with lesson materials they provided, and compiling a descriptive analysis (Creswell, 2013; Merriam, 1998; Yin, 2009). These methods are appropriate for this study, which seeks a rich, context-specific, and descriptive understanding of complex problems. Beyond description, however, the goal was to distill the subjects' perspectives into concepts that could be applied more generally. Therefore elements of grounded theory were employed in the analysis to connect and process ideas. Grounded theory methods generate understanding (theory) inductively by constantly comparing data collected in the field. They give primacy to the researcher's data over ideas gathered in the literature from other authors (Creswell, 2013; Glaser & Strauss, 2009).

Within the grounded theory school, Glaser (1978) and Charmaz (2014) offer more flexible methods, which emphasize the judgment of the researcher (the investigator's "theoretical sensitivity") while Strauss and Corbin (1990) prescribe highly structured procedures, particularly for the stages where initial ideas are connected and synthesized into integrated findings. Because of my background in modeling and for flexibility, I selected the procedures and logic of Glaser (1978) and Charmaz (2014): initial codes, focused codes, memo writing, sorting, drafting, and revising. My application of the theories of Brousseau (Brousseau, 1997; McShane Warfield, 2014) to the modeling literature, especially the Common Core modeling cycle (NGA & CCSSO, 2010), informed by my own teaching and modeling experience (Huson, 2015), served as a conceptual framework to support the collection and analysis of data from the field (Maxwell, 2012).

Interwoven process with literature review, interviews, and analysis overlapping. Concurrent data collection and analysis is one implication of using qualitative methods (Merriam, 1998), and grounded theory in particular (Charmaz, 2014; Glaser, 1978). That is, research activities are interwoven rather than sequential. The literature review, interview protocol development, subject selection, interviews, and analysis were activities that overlapped in time to support and guide each other in order to answer the research questions. The practice of constant comparison of the data from different sources uncovers linkages and eventually a synthesized interpretation. Glaser contrasts the method with "more conventional" steps of deriving hypotheses from the literature that are then tested against data. Instead, the goal is that "the theory is rooted in data not an existing body of theory" (1978, p. 38).

Steps in the research process. A brief recount of the sequence of activities illustrates how the research design elements overlapped in time and supported each other. My work began with an interest in modeling, then added a focus on the needs of teachers in the Common Core implementation (NGA & CCSSO, 2010). The study's purpose statement and the latter three research questions were developed as a research prospectus that was then successfully defended and approved (including a draft of the proposed Methodology chapter). I desired a theoretical component to the research, a framework to understand the teacher perspectives and give the findings rigor and wide application. In consultation with my advisor, the theories of Guy

Brousseau (1997) were selected to provide a conceptual framework for the study. Brousseau's ideas are highly applicable to the context of modeling instruction, and while respected abroad, they are not well known in the United States. Brousseau's theories have not been applied to the Common Core standard. Thus, this study fills a gap in the literature. As Maxwell advises, "The most productive conceptual frameworks are often those that integrate different approaches, lines of investigation, or theories that no one had previously connected" (2012, p. 35).

The first research question was added in order to include Brousseau's theory in the scope of the proposal, and, along with an interview protocol and disclaimers, my application to begin research was approved by the IRB. I conducted the first three interviews and transcribed the dialog from recordings. I then coded and began analyzing the data as I proceeded with the next three subjects. Using constant comparison across the data I derived concepts and categories linking the subject responses to Brousseau's theories and the modeling literature. As Glaser put it, the goal was that "this linkage, at minimum can place the generated theory within a body of existing theories. More often, ... it transcends part of it while integrating several extant theories. It may shed new perspectives and understandings on other theories and highlight their process" (1978, p. 38).

Developing interview materials. In order to collect specific and comparable data from the participants, modeling cycle and lesson examples were developed to use in the interviews. The rationale and features of those exhibits are presented in the following paragraphs. The documents, a Common Core excerpt and three modeling lessons, grew out of the literature review, my modeling and teaching experience, and the work for the teachers' guide Handbook III (Huson, 2015). Grounded theory considers the investigator's experience an important factor for gathering data and for its analysis. Glaser calls the capability "theoretical sensitivity," and he explains, “Most generally, the background experiences of one’s education and training is used to *sensitize* the researcher to address certain kinds of broad questions. … [forming] guidelines and reference points which the researcher uses to deductively formulate questions which may then elicit data that leads to inductive concepts being formulated later” (1978, p. 39).

The Common Core depiction of the modeling cycle (NGA & CCSSO, 2010) was used with teachers to elicit responses pertinent to research question two regarding the modeling cycle. The second exhibit was a modeling task used in a course I attended by Borromeo Ferri with Blum (titled “Bale of Straw,” Kaiser et al., 2010, see Appendix C). It was selected as a very simple case of modeling materials and one that invited comments regarding estimating and students’ comfort with ambiguity.

Two modeling lessons were taken from the mathematical modeling handbook series, in part because of the complementary subject matter they covered, in part because I was familiar with them. I had studied one in a professional development modeling course with Pollak (Gould, 2013), and I had also taught it to my own high school students. That lesson, “Narrow Corridor” (Tan, 2012), and the lesson “Sunrise Sunset” (DePeau, 2012) were both treated in my chapters of Handbook III. The first two publications in the modeling handbook series offered 26 modeling lessons and matching assessments. Handbook III continued with a subset of those lessons, developing “paradigms” to assist teachers to implement modeling in their classrooms. Thus for a particular two-day lesson, the first handbook contains student handouts, the second offers assessments, and Handbook III advises on strategies for teachers.

The Common Core modeling cycle is the organizing principle for preparation, instruction, and assessment that runs throughout Handbook III. Each of the coauthors developed special areas of interest. My Handbook III paradigm relates to outside resources and support, inspired by an expansive notion of Brousseau's milieu. A second theme is how to manage teaching difficulties, which Brousseau explained with his concepts of the didactic contract and learning obstacles. Suggestions for collaboration are included, another example of match of the publication with the scope of the study's research questions.

Data Collection Methods

Interviews were the primary method used to collect data for this qualitative study. Interviewing collects rich, dense descriptive data (Merriam, 1998; Yin, 2009). It allows subjects to explain their views, refer to examples, and confirm that they were clearly understood. I could also ask follow up questions to request detail to their response or to pursue promising ideas.

The interviews were conducted in person and they each lasted roughly an hour. At the beginning of each interview I made sure the subjects read and understood the IRB disclosure, giving them a copy and securing their consent and signature. I continued with a few questions regarding their experience, sources of instructional materials, and their familiarity with the Common Core standards. To standardize this background data collection and to ensure it was complete I used a printed template, which I filled out and retained. (The intake form is shown in Appendix A.)

I conducted my meetings with the subjects in a format Merriam calls a "semistructured interview" (1998, p. 74), Yin terms a "focused interview" (Yin, 2009, p. 107), and Charmaz calls "intensive interviewing." "Key characteristics of intensive interviewing include its: selection of research participants who have first-hand experience that fits the research topic; in-depth exploration of participants' experience and situation; reliance on open-ended questions; objective of obtaining detailed responses; emphasis on understanding the research participant's perspective, meanings, and experience; and practice of following up on unanticipated areas of inquiry, hints, and implicit views and accounts of actions" (2014, p. 56).

The discussion was open-ended and conversational, but it followed a protocol I had written to ensure that responses were collected to answer each research question (Appendix B). The prepared protocol also added an element of formality to the interview. Because I knew each of the subjects either personally or professionally, the formal structure added rigor to gathering the independent perspectives of the participants, uncolored by our prior connections. The protocol was written as a detailed list of questions but in fact was used as a flexible guide, serving to prepare me to gather rich data rather than act as a script. Charmaz describes an interview protocol an "interview guide" which she recommends serves "as a flexible tool ... [helping researchers] to think through the kinds of questions that can help them fulfill their research objectives" (2014, p. 62).

Hard copies of the Common Core exhibit and three modeling lessons were used to elicit and clarify subject responses. At times participants made notes on the documents, which I retained to be available during the analysis process. Participants frequently volunteered their own noteworthy modeling lessons. In several cases they gave me copies of the materials or of examples of their students' work. In one case, Alice, I conducted a second in-person interview specifically to review her lesson materials and follow up points raised in our first meeting. Triangulation is the practice of comparing different forms and sources of evidence to corroborate findings, as for example in the case of Alice's lesson artifacts, her original responses, and our follow-up (Creswell, 2013; Merriam, 1998; Yin, 2009).

With the subjects' permission, the discussions were recorded, and I then transcribed each interview. I found the process of listening to the recordings and typing them up time-consuming but fruitful. (There were 209 pages of transcripts.) I reflected on the interchanges and revised my conduct of the questioning (Bloomberg & Volpe, 2012). The transcribed dialog also gave me a rich and detailed set of data to complement the notes I had taken during the meetings. As Charmaz recommends, “Coding full interview transcriptions gives you ideas and understandings that you otherwise miss. ... Coding full transcriptions can bring you to a deeper level of understanding.” (2014, p. 136). The coding and analysis process is discussed next.

Data Analysis and Synthesis

The interview data were analyzed using methods from grounded theory (Charmaz, 2014; Glaser, 1978). I selected these methods of coding and memo writing because, in addition to a rich description of the teachers’ perspectives, I wanted to conceptualize their views into an analysis that would be relevant in other, similar contexts. As Charmaz explains, “Coding moves you toward fulfilling two criteria for completing a grounded theory analysis: fit and relevance. Your study fits the empirical world when you have constructed codes and developed them into categories that crystallize participants’ experience. It has relevance when you offer an incisive analytical framework that interprets what is happening and makes relationships between implicit processes and structures visible” (2014, p. 133). I used software to record excerpts of the data, codes, and descriptions and ideas in the form of memos. Using a specialized analytic database had practical advantages and added reliability to the study (Yin, 2009). My memo writing began early in the analysis of the transcripts and gradually evolved into draft notes for the Results chapter. “Memo-writing constitutes a crucial method in grounded theory because it prompts you to analyze your data and codes early in the research process” (2014, p. 162).

Pilot manual coding. A brief recounting of the evolution of the coding and memo writing process illustrates how general concepts arose with more explicative power to synthesize the teacher responses. I made a first pass over a paper copy of the first two transcripts, writing ideas and labels in the margin. This manual coding included many initial comments with little attempt to standardize them, but there were some abstract ideas as well. I followed Charmaz: "I advise coding everything early in your research and see where it takes you as you proceed. These early codes can lead you to identify focused codes quickly" (2014, p. 112). Merriam is even stronger in her advice to begin coding right away. "The right way to analyze data in a qualitative study is to do it simultaneously with data collection" (1998, p. 162). My first notes were then distilled into a small number of prototypical codes for entry into the analytic software package. These initial codes were similar to the ideas from the literature and my preliminary conceptual framework, ideas that also influenced my interview questions. Charmaz calls these ideas sensitizing concepts. "Sensitizing concepts can help you start to code with your data. These concepts give you starting points for initiating your analysis but do not determine its contents" (2014, p. 117).

Data analysis software selection. I selected a data analysis software package in consultation with the professor of my Qualitative Data Analysis course, Victoria Marsick. Software can make analysis more efficient (Creswell, 2013), but it adds a dimension to the research design (Richards & Morse, 2013). A database of the data collected and generated can also increase the reliability of a study (Yin, 2009). The interview transcripts were stored and analyzed using the online software service Dedoose. Dedoose is particularly good for navigating among transcript data, codes, categories, and free-form memos. Dedoose has multiple aggregating and quantifying functions, but in practice the most important capability for my work was simply generating links between data, codes, and memos in order to make comparisons, distinctions, and spark new ideas, in other words, the constant comparison method (Charmaz, 2014; Glaser, 1978).

First codes were a poor fit. When I applied my initial codes using the software across portions of the first two transcripts, the results were unsatisfactory. I felt that my subjects' responses were not matching the codes. I was "forcing" the data to fit my preconceptions, a danger Charmaz (2014) and Glaser (1978) warn against. In response I drafted a less abstract set of topical codes, which essentially categorized responses according to the research questions they corresponded to. Richards and Morse note that "novice researchers find *topic coding* the most accessible" and it is suitable "when a research design clearly addresses specific topics" (2013, p. 150). A few of the initial codes that seemed to fit the data best remained, now hierarchically grouped under the topic codes. A complete pass over two interviews with these codes worked better, although there were still many ambiguities and poor fits. Most promisingly, however, the process was very productive in terms of generating new ideas, noted as memos.

Peer coding and discussion. During the time I began analysis I benefited from my classmates' review of my research as they independently coded selections from my transcripts, left me comments in Dedoose, and discussed it as a class. Some of their feedback related to specific codes or data, other suggestions were to the interview process or research design. It was a great aid to work with a guest Dedoose expert to set up and get the analysis underway in the online system. I also took the opportunity to cross check different analysts' coding and found that generally the transcripts were coded in the same way by multiple analysts. Dedoose offers sophisticated capabilities to statistically quantify "intercoder agreement" (Creswell 2013, p. 253) as a measure of the reliability of a coding schema; however, I decided not to employ those features for this study. I expected to support my findings directly with subject quotations rather than with quantitative measures. Furthermore, I considered the coding process as a tool to gain understanding of the data and generate theory (Charmaz, 2014), rather than to derive a coding schema as an end in itself.

Final coding and memo writing. The subsequent analysis was a inductive process of refining the codes, adding transcripts, linking data in categories, and surfacing themes in memos by comparing participant responses (Bloomberg & Volpe, 2012; Merriam, 1998). “At first, you compare data with data to find similarities and differences” (Charmaz, 2014, p. 132). The ongoing literature review and developing conceptual framework helped me make sense of the data, but I was careful to ground my memos in the teachers’ statements. “Extant concepts must earn their way into the analysis” (Charmaz, 2014, p. 112; Glaser, 1978). I followed Charmaz’s advice to avoid overcomplicating the coding tree. She says, “Initial and focused coding will suffice for many projects.... Keep coding simple” (2014, p. 147). The links in Dedoose among the data selections, codes, and memos gradually thickened. As Richards and Morse put it, “Coding is *linking* rather than merely labeling” (2013, p. 154).

The memos documenting themes and categories became more explicative and the activity took a slightly different character. Charmaz calls this phase focused coding. “Focused coding is a significant step in organizing how you treat data and manage your emerging analysis. ... The move from initial to focused coding is often seamless. For most of my analyses, focused coding simply meant using certain initial codes that had more theoretical reach, direction, and centrality and treating them as the core of my nascent analysis” (2014, p. 141). Eventually the categories and themes became a dense framework spanning the teacher perspectives and addressing the research questions. It is hoped that the findings will be meaningful and relevant to those seeking to understand the Common Core modeling standard implementation.

Ethical Considerations, Trustworthiness, and Limitations

Ethical issues are important in research with human subjects. I took care to safeguard the interests and rights of the participants at all stages of the study. While the matters discussed with the teachers were not likely to cause them discomfort or harm, the research topics were explained to the subjects before soliciting their agreement to participate, and they were told they were free to end the interview at any time. Informed consent is a critical protection, which was obtained verbally and in writing from the participants before the interviews began, using approved IRB forms. Efforts were taken to protect the privacy and anonymity of the subjects. The subject and school names in the data files and report were changed, no student-specific references were collected, and the recordings and transcriptions of the interviews have been under my control at all times.

Notions of validity and reliability are different for a qualitative study than they are for quantitative research. Merriam (1998) recommends several strategies that were used in this study to ensure that findings are consistent and match the reality portrayed by the participants' perspectives. A multicase study design was used to collect data from six teachers allowing triangulation among sources and contexts. The lesson materials several participants shared with me represent a second data form to corroborate their verbal responses. Peer feedback and suggestions that followed from my Qualitative Data Analysis class's coding practice with my data was also helpful. The Dedoose analytics software database served as record of the analysis as it progressed, adding to the overall integrity of the research. The investigator is the primary instrument in qualitative research, and I have attempted to document my background and theoretical orientation, and to be transparent with the subjects regarding my intensions. Creswell (2013) lists these same strategies and also suggests that writing detailed, descriptive findings can allow the reader to form an independent opinion regarding transferability. To that end I have supported all of my findings with quotations directly from the participants in the Results chapter.

Summary of the Research Methodology

In summary, this chapter has given a detailed description of the study's methodology. A qualitative multicase study design was used to understand teachers' perspectives on the modeling cycle and their instruction in the context of the new Common Core standards. Six high school teachers were solicited using purposeful sampling, each with modeling experience but coming from a variety of geographic and school settings. Data were collected through in-person interviews, transcribed, and analyzed using qualitative methods with elements of grounded theory. A conceptual framework was developed through a review of the literature and used for the interpretation the data. Themes emerged from the analysis, which were synthesized with supporting participant quotations for the results of the study. Conclusions were drawn and recommendations made for further research and the development of supports for modeling instruction. The intention is to add to our understanding of Common Core modeling standard implementation and to be useful to mathematics education authors, publishers, and supervisors.

Chapter IV: Results and Discussion

The results of the study are organized in sections addressing each of the four research questions in turn. A short summary introduces the findings to each question, and then, as much as possible, the subjects' own words are quoted to provide the detail.

Research Question #1: Brousseau and Modeling Pedagogy

This study's first research question is *How well does Brousseau's theory of didactical situations align with our current understanding of the modeling cycle?*

Brousseau's theory is a broad educational framework, whereas the modeling cycle refers to a specific set of mathematical practices and content. In part, the question of the alignment of the two is addressed in the literature review chapter. This section contains an interpretation of the subjects' experiences teaching modeling in the terms of Brousseau's theory, showing how it can be applied to modeling and how it can illuminate the phenomena the teachers recount. Three themes that emerged in the analysis group the perspectives into subsections. The first is how modeling promotes student engagement and related social factors that Brousseau termed the milieu. The second theme is that applying mathematics to real world problems helps students understand mathematics conceptually, a view consistent with Brousseau's general constructivism and specifically with his distinction of didactical versus adidactical problems. Third are the obstacles and difficulties teachers say students encounter when modeling because of their expectations and prior preparation, what Brousseau calls the didactic contract.

Engagement, Brousseau's milieu, and authentic modeling problems. Structuring a lesson as a modeling problem is a way to hook students' interest. Fred suggests, "If you do more modeling you can make these interesting problems, related to some TV show or some sports things. It grabs their attention a little bit more." For example, "Does it make sense for you to use a daily metro card or buy a monthly? When does it make sense? That is something that actually applies to their life and could save them money. What sort of loan would you take out? Those are real things that would help them if they had the math to do it."

The subjects of the study mention engagement as a key benefit of modeling work. They say that students can draw on their personal experience when formulating and working with their model and that real life problems motivate students. Alice cites the combination when she analyzes the "Sunrise Sunset" lesson. "You have enough clues from your own experience of the sun. Think about that.... I don't know that anyone cares a tremendous amount about when the sun rises and sets, but it is at least a real phenomenon that they have experienced in their real lives. And you could come up with people who would care... like farmers." She returns repeatedly to the motivational effect of the right problem. Regarding the "Narrow Corridor" lesson's question of whether a sofa will fit around a hallway corner, she says, "The kids would be curious. Is the couch going to fit? It's not a life or death situation. They're not going to care a lot, but they will want to know."

Brousseau termed problems that students believe have legitimate outside value as adidactical situations, distinguished from didactical situations, which are only relevant to school. He put adidactical situations at the core of his instruction. This parallels the Common Core authors' call for realistic problems (NGA & CCSSO, 2010) and what modeling researchers call authenticity (Galbraith, 2007; Vos, 2011). Students naturally feel able to take an active role solving authentic problems, and they recognize that the problems have a purpose in real life. Brousseau stressed how these types of problems play themselves out among the students and the classroom social factors involved, the milieu. As the subjects of this study explain, their modeling instruction often focuses on creating the right classroom dynamics.

Here is Denise describing how she introduces a geometry concept though a modeling lesson. “For example, outside my classroom you can see Lincoln High School. So when I walk in the classroom, instead of talking about the angle of elevation, I walked to the window, I opened the window, and I said, ‘What’s in front of us?’ They said, ‘Oh. Lincoln High School.’ All of the kids suddenly became very interested. … My second question was, ‘Have you guys ever wondered how tall Lincoln High School is?’ All of the kids became so interested. They all looked around me and they threw out their answers….All of the kids are engaged….More discussion. The kids were laughing. They were focused on the problem.”

Like Denise, Alice notes the importance of student engagement, and in her suggestions for the “Narrow Corridor” lesson she focuses on classroom dynamics and motivation. The problem is still to fit a couch through a given hallway, but the students pick from a variety of alternative sofas. “Give them a selection of couches, and they pick the couch. ... different colors, ... different dimensions, like long skinny ones and shorter, fat ones. So it’s not every single kid doing the exact same calculation. So then your question is, ‘What’s the best couch you can get in?’” The milieu’s social component is enhanced; comparisons among students can be made. To each student, it is their couch, and theirs is the best. Alice suggests, “That would add a little bit of investment.”

Brousseau’s lesson situations wove mathematics problems into the social interaction of the students. Mathematical properties emerged or were discovered by the students as they played a game or competed. Alice’s extensions to the “Narrow Corridor” problem enhance the social factors that Brousseau suggested we target, and highlight this potential in real world problems. She continues, “If you have different kids doing different things and it’s not immediately obvious which one’s going to work then I think they’ll care a bit more, or they’ll be frustrated when their couch doesn't fit. In a way, that kind of is analogous to the way you would actually be frustrated if your couch would not fit in the hallway." Regarding student interaction, Alice says, "Especially if you can get the kids to kind of argue. If some of the kids think that, 'No. That is a ridiculous choice. Why would you ever choose that couch? My couch is going to make it.'"

Note that Alice's modifications could also be used to differentiate the mathematics in terms of difficulty. One couch's dimensions may easily make it around the corner. Another may require a challenging three-dimensional geometry as the couch is lifted and twisted through the hallway. Students could then be assigned more or less difficult sofas at the teacher's discretion.

Notwithstanding the motivational benefits of the milieu, Brousseau warns that classroom conventions can also sometimes undermine the development of students' independent mathematical judgment (citation). Students are attuned to clues from the teacher and other school context, and inadvertent signals can replace student problem solving. Teachers report exactly this phenomenon; depending on how modeling activities are scheduled and how students are prepared and practice, it can be difficult to prevent an authentic modeling problem from losing its novelty and openness. For example, Alice comments about the "Sunrise Sunset" problem, which employs a periodic sinusoidal function, "Well, they're probably in the middle or the end of a unit on periodic functions, so a lot of students would use their context clues, not from the problem but from school. Say, 'Well it's going to be a periodic function.'" The benefits Brousseau sought from adidactical situations and that modelers seek from authentic problems disappear if instead of using the problem's requirements to drive the mathematization, students anticipate the course sequence and assume the techniques most recently taught in their class will provide the solution.

Brousseau's adidactical situations and giving mathematics meaning. Brousseau designed problem situations that naturally require students to develop desired mathematical methods in their solution. Students' conceptual understanding arises from the problem context, the setting from which the mathematics is elicited. Brousseau says that the meaning of the mathematics derives from the context of application. Similarly, the modeling cycle starts with a problem situation and students must identify salient features of the problem and mathematize them. Subjects of this study report that suitable modeling problems do elicit mathematical concepts as Brousseau suggested, and that, moreover, learning is facilitated by students' personal experience with the problem's real world context. This is a key benefit of modeling problems: Modeling effectively develops conceptual understanding because students have personal experience to draw on. The immediate real world of the classroom, the milieu, is a particularly effective context in which to situate problems.

This section begins with examples of classroom modeling situations, moves to participants' views about how conceptual understanding is elicited by modeling problems, and finishes with the issue of time constraints.

Bringing the physical system to be modeled into the classroom allows the teacher and students to directly observe and manipulate it. Belle uses students themselves as the real life actors, using a motion detector and graphing software to plot their position and speed as they walk in the front of the classroom. Students then interpret the resulting graph. Her description is exactly how Brousseau would say conceptual understanding naturally arises. "I do this the first day of class, so they can see that when the kid's walking real fast the line's steep, and when he stops—so it's a time versus distance—when he stopped it's flat. When he's coming toward me it's down, then he stops, then goes backwards. And the slope of the line relates to the speed at which the kid is moving."

Carol describes how she puts a student in a wheeled chair with ropes tugging it in various directions to introduce mathematical vectors. "I'll do a piece on vectors. So trying to help the students understand what exactly is a vector, and how do you work with them, and how do you use them to model real world phenomena. So I'll put a kid in a wheelie chair, an office chair, and we'll tie ropes to it. One person pulls and then two people pull. How will we then translate that into the concept of vector to represent what was happening with the force and the movement, the displacement."

Problems are brought to life by actions the teacher takes to make them real life. Moving furniture or a box in the classroom would prepare students for the "Narrow Corridor" problem, Carol suggests. If not physically moving it—which would be best—then remembering and discussing a time when they did move a piece of furniture. Carol says, "So you bring the experience first and then bring [the problem]." Start by "looking at the physical constraints of the world and trying to figure out, and then move into [the 'Narrow Corridor' handout]." The rationale is not only motivation and engagement, it is also conceptual. "Also you understand it differently," she says. This is the sense that Brousseau distinguishes adidactical situations, how they give significance and meaning to a mathematical concept. Geometric knowledge born of a student's experience of "the physical constraints of the world and trying to figure it out" (quoting Carol) will take on richer meaning and significance than knowledge from listening to a teacher.

Fred also sees the benefit of application when learning mathematics, taking meaning from context. "I think it helps them to understand, like what does a linear function mean. It's going at a constant rate. What does positive and negative slope mean? … If there's some kind of context to that I think it makes more sense. What does it mean that it's a root? Oh, it means that your *y* value is zero, but in the context of some other problem it means that there's no money, or the income and expenses is a breakeven, or something. Some physics problem zero means it hit the ground." He remembers himself as a learner. "I think modeling in math is important because it helps to give it a context. It helps you understand it better…. When I think about my math education there was times when things were really kind of abstract and then once I applied it to the situation it made a lot more sense. I think it's very important."

After describing how she introduces the problem mentioned above to measure the height of the Lincoln High School building next door, Belle goes on to explain how a concrete, local problem like this gives all students an entry point to the mathematics. "All the kids are very, very excited, because, first of all, it's something related to their real life. Second, they know how to do the estimation; it's also related to real life. Third, it's very, very visual, even though there are terminologies involved, it is very clearly demonstrated to them. So they can clearly understand the concept without much language or anything that sounds very challenging to them. So all of their focus is on exploring how to solve this problem. It's a real life situation, and it's very, very obvious to them." English language learners in particular benefit, she says.

Their own common sense experience of a real life situation provides students with an independent basis to evaluate a mathematical solution (Niss et al., 2007). This is the essence of Brousseau's adidactic situations: Students believe the solution lies in their hands rather than being based on the teacher's officially correct opinion. As Alice recommends, "I try very hard not to be the person giving the right answer." In a related way, the Common Core requires that students be able to make "plausible arguments that take into account the context from which the data arose" (Mathematical practice standard #3, NGA & CCSSO 2010, p. 6). Several of the teachers cite students' independent sense of agency as a benefit of modeling problems.

For example, Fred, in his criteria for a good problem, includes whether students have enough background to judge the validity of their solution. "The ideal problem would be interesting to them. It would be at the right level for them mathematically. They would be interested in finding out the right answer for some reason other than you just asked them to. And their answer…They would be able to understand from the result… They would have ways to know whether or not their answers were kind of right or in the ballpark." Again referring to himself as a student and how helpful it was to start with a personal experience, he says, "I could picture it in my head. It really helped me to make more sense of it."

Alice is explicit with her students: that for modeling they will be expected to use their life experience to check their mathematical results. "Once it was introduced to them as a concept—'Don't check your common sense at the door when you walk into a math classroom. The real world still applies.'—I think it was actually helpful for them in helping to frame their answers, or give them confidence when they got an answer that was within the realm of reality, that they were right. So once we introduced that common sense aspect of it I think it actually helps."

Denise points out the sense of agency that arises because modeling problems are grounded in real life instead of school. "They feel it's more real. They feel they are really solving this problem, instead of the teacher telling them."

When the subjects of the study explain constraints on their modeling instruction they refer to tension between a student-inquiry approach versus direct instruction (Burkhardt, 1989; Pollak, 1969), in other words whether or not to practice constructivist methods. Modeling and the Common Core they identify as constructivist (as would be Brousseau's theories), while high-stakes testing, raised performance standards, and the accountability movement are pressures toward direct instruction methods.

Is modeling necessarily conducted as a student investigation? Belle recognizes that the ‘Narrow Corridor’ lesson has students discover their own solutions, and she laments the class time necessary: “I love to do things like this. The main frustration that we’ve found is the time constraints. … With the amount of material we’re supposed to cover, a lot of teachers don’t feel like they have the luxury of spending this amount of time on these investigation-type activities.” But she says she believes student retention is better when learned in this manner. “If they discover it on their own, they will retain it. They will learn it. But if the teacher presents it to them, then their retention is much, much lower.”

Ellie echoes the sentiment that time is a driving constraint. “The issue is of course, time. A lot of times the kids will have lots of really interesting and really great ideas, but they won’t get to the one that you want them to get to, or they will but it will take 30 minutes. So there is this tension between how much do you really let them think, and how much do you have to push them along.”

There may be less time consuming ways to gain some of the benefits of real world problems. Carol says she frequently has students dream up situations themselves (Galbraith et al., 2010). “We don’t necessarily have to solve them,” just plant a seed. She says it makes the students “break out of some fixed lines of ‘this is the only way to do math.’ And sometimes this is enough. You don’t have to really spend a lot of time but it’s the dispensation that you have about your relationship towards math, and what is it? Is it fixed? Or is it alive?” Students may hold a limiting belief that “this is the only way to do math,” a phenomenon Brousseau called the didactic contract, and they will object when teachers challenge that. Redefining students’ views of mathematics as not just correct, but “sensible, useful, and worthwhile” (NGA & CCSSO, 2010, p. 6) is important according to Carol’s Waldorf philosophy. Mathematical modeling is a way to understand the world. “This is about thinking about the world in your own way, and understanding in your own way, and really looking at things differently, is what we’re training them to do.” She is quite expansive. “We want them to be engaged in the world for that’s sake, so that they come out being adults who are interested in the world around them and want to make a difference and find their path in life.”

Brousseau’s didactic contract and problems in modeling instruction. Successful mathematics students have mastered many procedures by high school. Knowing a right approach and correct implementation is inherent in their view of doing mathematics. Modeling problems challenge those prior conventions. Brousseau identified a dynamic he called the didactic contract, that students expect teachers to show students the “right” way to do mathematics. Sometimes new mathematics requires that the contract be renegotiated. The subjects of the study say students do object to certain modeling practices, citing the need to estimate model inputs and to rework invalid models as two specific requirements students resist.

For example, Fred’s says this about the “Bale of Straw” problem (Kaiser et al., 2010, see Appendix C), which requires students to estimate quantities, “Most classrooms they just expect, we’re going to give you the numbers that you use to solve the problem. You take these numbers, do something with them, and you get the answer. And if you don’t give that to them, they are going to be kind of…It’s a little shocking for them, I would say.”

The Common Core modeling standard calls for students to be “comfortable making assumptions and approximations” (NGA & CCSSO, 2010, p. 7; see also Pollak, 1969). Teachers say modeling is a good way to teach estimation, but it requires resetting expectations. Ellie says, regarding the “Bale of Straw” modeling task (2010, see Appendix C), “I think this is a great example … where you’ve got a very simple image, no actual numerical data at all, and it’s up to the students to figure out how the heck we would answer this problem based on what they have shown us. I think this is a good problem.” But students will push back against the lack of firm numbers to base their calculations, “I think I’d have a lot of students who would say, ‘How?’ Or, ‘OK. How tall is it?’ They would immediately look to me to give them more information.” She goes on to explain how she would scaffold and differentiate among her students so they understand the problem and formulate the model without giving up. “So the idea of this problem is, absolutely anyone could start on it, the thing is, not absolutely anyone is going to start on it.”

Alice too notes the challenge with approximation. In fact, she strongly identifies with the students. “If I were a student in this class I would absolutely hate this [the “Bale of Straw” problem (2010, see Appendix C)], because there’s not going to be an exact, right answer. And that’s the kind of student I am. As a teacher, I do like it because it forces estimation. And it forces you to make educated guesses about how tall that person probably is. It’s a good estimation exercise. But I would be very frustrated at the end, to not know the right answer.” How to teach students to make assumptions and reasonable approximations is an open question. “Estimation is something we’ve talked about as a math department. It’s something our students struggle with tremendously.... I don’t think estimation is taught explicitly hardly at all, at least not in the high school level. I don’t really know how to teach estimation. I would like to teach it better.”

Belle’s experience using mathematics professionally makes her very aware of the need to make assumptions and estimate model inputs. She recalls a popular problem addressing estimation. “Suppose you’re stuck in a traffic jam on the highway; it’s at a standstill. And you know you're exactly one mile from your exit. How many cars are between you and the exit?" It is simple and concrete, essentially being a series of estimations. "It's a good problem because the kids will say, 'I don't have enough information to do that.' But you do, because you just have to assume, well the average car is 15 feet, and there's an average of 4 feet between each car, so that's 19 feet. 5280 feet in a mile divided by 19 feet. There must be this many cars, times four lanes."

Another modeling practice that triggers student resistance is the potential need to rework an initial solution. The Common Core modeling standard says students must realize that their assumptions "may need revision later" and that they will possibly need to improve "the model if it has not served its purpose" (NGA & CCSSO, 2010, p. 7). Students have learned, however, that they should solve a problem the right way the first time. As Ellie explains, multiple iterations and invalid models violate student expectations. "The issue is that students are more comfortable going from problem directly to compute and then report. So they don't… There is going to be a lot of resistance to this idea that, 'Wait, you want me to work on an answer that might not be right? Why would I waste my time doing that? Just tell me what the right answer is.'" What is required is more than teaching a new skill, model validation; it is renegotiating the didactic contract, which is not easy. "They are very used to how they think the classroom is supposed to work. And if you try to ignore that and work around it, they'll think that you are trying not to teach them, or that you're a bad teacher. There'll be a lot of resistance around that….They really know what school is supposed to look like and if you deviate from that, you are the problem, not them."

Ellie explains exactly how this impacts modeling cycle instruction. "I can tell you that this arrow right here, the one that looks like it is going backwards [iterate the model after validating], that's the one they don't want to do. That feels like going backwards. When it feels like you've got an answer and then you find out that it's wrong and you have to start over, that's very defeating. … You're supposed to start and do the right work and get the right answer and then you're done and you don't have to do any more math. Instead, there is a process that actually…. It's a cycle. You go through it and you come back to the beginning, but it's not the beginning. You've still made progress." The goal is for students to do this independently. She says, "The issue with modeling is...they need to be able to think about all this themselves without the teacher jumping in and telling them, 'Do this, do that, do this. Good. Now you got it.'"

Summary of results for research question #1: Brousseau and modeling. The participants do not reference Brousseau explicitly, but their views on several topics imply that Brousseau's theories are particularly applicable to modeling instruction. The benefits of modeling in terms of student engagement, the classroom dynamics teachers employ to animate interest, and the threats to authentic problems from too much practice are all phenomena explained by Brousseau in terms of the milieu. Teachers prize real world situations to promote mathematical understanding, and the way they ground problems in the students' lives, develop a sense of independent agency, and try to teach modeling through student inquiry are well aligned with Brousseau's methods using adidactic situations to create mathematical meaning for students. Furthermore, the idea of the didactic contract explains issues that teachers cite as they expand students' conception of what is the proper practice and application of mathematics.

Research Question #2: The Modeling Cycle

This study's second research question is *To what extent do teachers consider the modeling cycle as an important framework to structure their instruction? How, if at all, do they report it influences the way they prepare to teach mathematical modeling and assess students?*

As a framework, the modeling cycle fits with the participants' views of how students do modeling. Their experience teaching modeling, however, uncovers issues with each of the modeling steps as well as an overall need for more detail and structure. This section is organized by considering the teachers' perspectives on the cycle as a whole first and then regarding specific steps. The first half of the cycle, understand the problem and formulate a model, is treated as a unit, and then the interpret and report phases are discussed. The conclusion offers further detail regarding the validity test and the iterative nature of the cycle, which was introduced above in the section on student resistance and Brousseau's didactic contract.

The cycle as a useful framework but requiring additional detail and structure. The Common Core modeling cycle is an accurate high-level framework. Teachers report it as broadly consistent with their practice. For example, Alice says, as she refers to the diagram, "I don't use these words, but this is very similar to how we talked about modeling this year." At her school "decontextualize" and "contextualize" replace formulate and interpret. (The Common Core uses these two terms in its Mathematical Practice standard #2, Reason abstractly and quantitatively, p. 6).

Carol agrees and suggests that the cycle is also a useful guide for students, although she would elicit it from them. "It's a good tool for the student [the modeling cycle], but I would prefer that they generate it, rather than it's given to them." She would highlight the validate step. "I think it's good because… what happens is they do the understand, formulate, compute, and think they're done. And so to help them, from their experience you say, 'Well wait a minute. What do you do next?' 'How do we know if it's right?'"

Ellie suggests that the modeling cycle diagram could be posted on the wall to help students understand the process and also to add credibility to the teacher's arguments for the non-linearity of the cycle, the need to validate and perhaps repeat phases. If renegotiating the didactic contract with students is a challenge, the authority of an official diagram may help. As Ellie puts it, "It looks official. Therefore you must be more official." However, below her general agreement with the official modeling cycle lies the caveat that quite a bit more structure and detail is required in practice. Ellie refers to her experience with urban, public school students: "The discourse around modeling seems to be, 'Oh you know, just give the kids an interesting problem and they'll get interested. And they'll go off and they'll do math and [have] ideas and questions.'" But that does not work. She continues, "Our students are a lot more hesitant than that. They need a lot more help than that." To work for her students, the modeling cycle must include more. "They are used to a lot more structure, a lot more prompting. They need to have a lot of scaffolding—right, that's the word—before they can really get started on the problem."

Other teachers echo the difficulty of trying to master the whole modeling cycle at once. They explain how they add structure by breaking down the steps into smaller tasks. Denise, for example, says, "This is a very high-level diagram to show the skeleton of math modeling. … You won't need much more than this simple diagram to develop your lesson, to design your teaching." But it requires a lot of practice for students to learn, over all four years of high school, she says. "Math modeling is definitely not one lesson. … not only one year. [As] kids go through the entire secondary education, or even college education, they need to keep practicing the math modeling." As modeling competency is built through practice and experience, at times attention must be focused on specific points in the cycle and their component tasks. She offers the first phase, understanding the problem and mathematizing it, as an example. "They will need to actually learn—for example in the very beginning when defining the problem, before formulate—they need to collect the data, and they need to validate the data." The next section explains other teacher perspectives on the first half of the cycle.

Structuring the problem and the formulation steps. The first two modeling steps are to understand the problem and formulate it mathematically. The Common Core modeling standard defines the problem and formulate steps as "(1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables" (NGA & CCSSO, 2010, p. 72).

Teachers find the Common Core cycle an accurate portrayal, but from their experiences teaching they add a rich set of responses, specifically on scaffolding, differentiation, and instructional tradeoffs. This is perhaps the most challenging stage of the modeling cycle. Several of the teachers explain how to structure this first stage, isolate, and decompose it. For example, they may guide it with a diagram, a data input sheet, or a given algebraic function. The appropriateness of such aid requires judgment and tradeoffs.

When asked about the difficulty of formulating a model, Ellie responds that that mathematical ability is "something that I feel like most people don't have. I would agree that that is the hardest part." She argues for breaking out data collection and formatting to initiate students into those sub-steps of the modeling practice, but it depends on her instructional goals and time constraints. "It's a good way to reinforce a skill that they need to have: getting information off of a website, which some of them may not practice. Things like putting numbers into a table. It's the kind of thing that I really want to be basic, but it's not going to be basic unless they do it over and over and over again. So I think that would be a good step depending on how much time you have for it." Responding specifically to the input tables in the "Narrow

Corridor" lesson, she says, "I think it is important to have the plot, the actual numbers, the actual coordinates. It's something, I don't know, it depends on what you want to the kids to ultimately be able to do. … In terms of what I teach in 11th grade, I think it's good to have that, the table there." They may not be formulating a model themselves, but they will be guided through the details until eventually they can model independently. "It's one of those multiple representation things. … It builds intuition."

Ellie's words are consistent with the Common Core standard: "Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity" (2010, p. 72). Carol echoes the creative sentiment, "They have to actually draw out and bring the artistic element to their charts and graphs, and measure." Commenting about the middle school curriculum, she says an important skill is composing "the whole visual layout."

The scaffolding included in the "Narrow Corridor" lesson may give too much help. Ellie says, "This diagram here, where you have the pipe in the corridor, is already starting to model the couch mathematically. Even picturing the corridor with this diagram … That's already mathifying the problem. … You've already figured out the important parts. Any time you can make a simplifying diagram of something in the real world, you've already said, 'OK. This is important, that's not important.'"

Alice also protests that the "Narrow Corridor" solution is given away by the inclusion of an input sheet that structures the approach to the problem. "It looks like it gives them, it tells them what to do." But then she asks about the students' abilities, "What grade is this for?" "High school geometry? I wouldn't give them this [the pre-formatted input table]. This is the thinking part. If you're giving them this then you don't need the whole story about the sofa and a hallway. Everything after here is just calculation." In summary, "If you're making the picture for them then you're doing the, I guess, the formulate step for the kids."

The grounding in student ability suggests that varying versions of the lesson materials might allow teachers to differentiate among students. Those who work faster, persevere, or have a tolerance for unstructured directions would not need the scaffolding given to weaker students. For example, Alice notes that the "Sunrise Sunset" instructional materials give students the functional form of the mathematical model. In this case she approves of that based on the needs of her students. "Then it gives them the model, though. That's what I was looking for. It gives them the equation, which is fine, because I don't think, at least my students haven't learned how to write a sine function to fit a scatter plot."

Denise offers a slightly different emphasis in her thoughts on data gathering. Students should learn that the quality of data must be assessed and justified. Regarding sunrise and sunset times, which are easily available online, she comments, "For many of the students, their first response is to go online to search for the answers in the beginning, but then the question itself actually asks them to think behind the answer, to provide justification for the online answers. Also, while the kids work through the exercise, the math modeling process, they will learn to assess the answers they find online."

Managing time constraints is a practical concern that Denise and the other teachers cite repeatedly. Asked whether a formatted set of times should be included in the materials provided (as they are in the "Sunrise Sunset" lesson version) or whether the students should collect the data themselves, Denise answers, "It depends on the time constraint. If I have plenty of time, … I would ask the kids based on where they live, … to go online and search for their sunset time and sunrise time, and based on their findings to answer the questions." Alice also puts practice with data collection and organization in a gray area in terms of priority. Answering whether it is important that students be capable of collecting information she replies, "I think it definitely matters. But I wouldn't say it matters as much as a lot of the other skills. I know that I often feel crunched for time. On a long list of standards that need to get covered, going out and finding their own data is not really something that I would prioritize."

Recontextualize: the interpret, validate, and report steps. Once the mathematics have been formulated and solved, the second half of the modeling cycle involves making sense of the results. The Common Core modeling practice standard states that students "routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose" (NGA & CCSSO 2010, p. 7). The modeling cycle diagram details three steps, interpret, validate, and report. The meaning in context of the mathematics is at the core of each. The study's subjects report that making mathematics meaningful is both a high priority and a key benefit of modeling. That makes this part of the cycle a natural starting place to introduce students to modeling. Furthermore, teachers tend to treat these three steps together. Teachers may be saying that while in principle the interpret, validate, and report steps can addressed separately, in practice students accomplish them together. In contrast with the first steps in modeling—decontextualization—which they suggest benefit from practice as smaller, incremental tasks, the latter half of the cycle is best taught and practiced holistically. Making meaning is the common activity, the essence of the contextualization half of the modeling cycle.

Fred repeatedly highlights the benefit of real world application for mathematical understanding. He believes students give meaning to mathematics through the context of its application (a view at the center of Brousseau's constructivism as well). Therefore students should start by interpreting mathematical models given them. "I think in the earlier stages I think they should practice with interpreting models." The more difficult formulate stage is taught later. "As they progress they would be developing a model and using it to predict something or whatever."

Belle also focuses students on the meaning of the mathematics in terms of the problem situation. "I emphasize that a lot too. Like in a linear model $ax+b$,…when I test them, or do anything, I say, 'Explain the meaning of the a,' and 'What does it mean in this context?'" Interpreting numbers is different from calculating them. That's "the hardest thing for some of them, is to interpret the meaning of the various coefficients of the model....I guess it's because they're so used to doing math as just computation, numbers, numbers, numbers, and they haven't really thought about what those numbers might mean in a real life situation, that that number actually has meaning." "That's the connection between the math and the real world." Furthermore, interpretation includes connecting meaning across multiple representations. Belle says, "So you're really relating the numerical, the Cartesian coordinate plane, and the real life situation. So there's really three pieces you have to put together: the equation, the graph, and the situation." (Later she calls it "the verbal description of the situation.")

Alice has extensive experience teaching the contextualize half of the modeling cycle. She says the students should "interact" with the mathematics. Noting an example from the "Sunrise Sunset" lesson where students give a personal interpretation of the model, the daylight hours on their own birthday, she says, "They have to interact with the scatterplot, which is good, and estimate—like your birthdate isn't a data point. So, I like these questions."

Alice's school (an urban charter school with largely low-income students) holds special "oral" proficiency examinations, an assessment that for mathematics involves the solution of a modeling problem and the presentation of the result to a small committee of teachers and a parent. The school has developed a set of problems and a routine of preparation over a number of years. Plenty of time is given for work on these problems on a regular schedule. A mathematical function to model the situation is given to the students, bypassing the formulate step, and therefore the focus is on the latter half of the modeling cycle: interpret, validate, and report. "The main purpose of the Orals is to see what they can do with the function, and not to see whether they can write an equation," Alice explains. Students must interpret various mathematical representations, algebraic, graphical, numeric, and they must interpret the real world meaning of the mathematical results.

"I think that this is asking them to pull together things that they know. So for example, the quadratic one requires them to know graphical solutions, algebraic solutions, but also understanding concepts like what a break-even point would look like graphically, what a break-even point would mean for the equation, but also understand the concept of breaking even." It is the connection the student must make between the mathematics and the problem situation. Alice says, "Right. Your profit is zero. That means the equation is equal to zero. And making that leap from what you know about the world to what you know about the equation."

The interpretation, validation, and reporting steps are assessed together. "My favorite part of the oral proficiency is the questions at the end. Where you can ask students, 'How did you know that the *x*-coordinate of the vertex was the answer to part B?' or 'Why did you ignore the intersection to the left?' And I think that's the most valuable part of that, where they explain the thinking behind their work." Alice continues, "I put, personally, a big emphasis on logical reasoning and explanation. I think that is one of the biggest things kids can get out of a high school math course: is being able to think in a reasonable way, and then convey that thinking…. They have to explain what they were thinking."

Alice's school's highly developed modeling assessment practice demonstrates why the Common Core cycle's steps to interpret, validate, and report are naturally integrated. Logically one can separate these three actions, but in practice one cannot evaluate a model without understanding the meaning of the mathematics in the real world context, and the reasoning must be explicit. Articulating a clear explanation—reporting—is necessary to demonstrate the prior steps. She says, "We just ask them to explain how they got their answer. That is the language we would use. We talk to them about being really explicit about their whole thought process, and why they are doing the things they are doing."

Alice believes explaining one's thinking—the focus of the school's oral examination practice—is a valuable and transferrable skill. She opines, "I think both the checking to make sure your answer makes sense, and also logically explaining your process, are the most transferable skills. And I mean transferable not just across math classes, but across content areas…. The skill of being able to explain what you are doing and why you are doing it, is the most critical thing."

As an aside, Alice says that sometimes students have too much practice, so that instead of first examining the real life problem to determine a mathematical approach, the student goes straight to giving an expected mathematical answer. In that case the real world authenticity of the problem is lost (Niss et al., 2007). Brousseau (1997) would say that the students are accustomed to finding patterns in school problems and that it would be difficult to maintain a supply of adidactical situations if the practice is too routine and thorough.

Validation is only practiced in a limited way. The study's subjects repeatedly bring up examples of checking the reasonableness of a calculated result. "Does it even make sense that this number is negative?" is a question Fred suggests a student should ask. The validation step specified in the Common Core modeling cycle is much more than this, however. With a wealth of modeling experience from a previous career, Belle is well aware that having students check their calculations is not the broad assessment of the overall applicability of the mathematical solution to the real world problem that is really required. Referring to her students, she says, "Because they're used to, that if you do the math correct then they are correct. But here you have the whole issue of fit. Does it really reflect the real world situation?" The subjects of this study indicate that validation in school modeling is more concerned with checking the mathematics than confirming the understanding of the problem and assumptions made in the formulate stage. Teachers say it is important for students to check their results, but they do it in a limited way, and furthermore it rarely leads to a completely new iteration of formulation, contrary to the modeling cycle and adult practice of modeling.

Students do get a sense of validation, rejecting values that would not make sense in a real world context, but it is limited. The most common example is a negative number. Alice explains, "Frequently we'll have a problem where it works out that one of the solutions doesn't work with the real world context. For example, the price would be negative 13 dollars. So the students would then need to explain why they have selected the one root instead of the other root."

A common modeling lesson is to apply a quadratic function to either a price-profit context or to a falling body. Referring to one of her lessons, she says, "This one is the price-profit, so it is breakeven. The other kind of quadratic is generally a kind of projectile, and it's, 'When does it hit the ground?' (the zeroes)." She addresses the problem of negative numbers early in the school year, asking her students, "'Well, your answer is that the ball hits the ground in negative seven seconds. Let's talk about that.'"

Alice's comments show the sophistication of the modeling practice at her school and the focus on interpretation and validation. "The validate step—does this make sense in real life—was something we really emphasized. … A major unit was quadratics so we had price/profit graphs and we had a lot of 'You hit a baseball, how far does it go?' kind of, real world situations. And we talked a lot about negative distance, negative time, negative price: why those things wouldn't make sense. … It was more important that they understand the big conceptual things like that there's no such thing as negative distance than that they plugged into the quadratic formula exactly perfectly."

A generalization of the requirement to reject negative values is to consider the domain over which a model is applicable to the real world situation. Interpreting and relating a function's domain in the context it models is a Common Core standard (F-IF.5, p 69). Belle mentions this opportunity to reinforce students' understanding of domain in validation. "There're some real world constraints. So that's a good place to bring in the whole idea of domain and range." In this case, Belle positions modeling in service of her goals to teach context standards, to make the mathematics meaningful. Alice expresses a similar opinion regarding function domain.

Overall, the participants' comments suggest a shortfall in practice versus the Common Core modeling cycle's call for a robust validation of results against the real world context. The actual practice in schools is limited to assessing the mathematical calculations. Conversely, what qualifies in modeling as an appropriate solution may affront the precision students are accustomed to requiring for school problems. This can be frustrating for students and teachers. Alice reflects on her own experience solving a modeling problem in a professional development setting, “I remember being incredibly frustrated after we’d worked on it for like an hour, and we had a bunch of answers … all clustered around some reasonable number. Nobody was way off or different. But I was incredibly frustrated that there wasn’t an answer at the end.” The weak treatment of validation may extend to another facet of the modeling cycle that teachers find troublesome, the practice of reworking a model found deficient.

Recycling through the steps and a dynamic modeling process. A key feature of the Common Core modeling cycle is its iterative nature. The decision box, validate, in the diagram may lead to a return to reformulate the mathematical model if the initial solution inadequately addresses the real world problem. Teacher responses vary as to whether this accurately reflects practice in schools. On the one hand, Carol agrees with the Common Core’s cyclical framework, even suggesting that the standards diagram might imply a step-by-step linearity to the process that in fact was much more fluid. In contrast, other teachers point out that for routine modeling problems, which are those most frequently practiced and assessed, students do follow the diagram in a linear way, and the validate step is limited to a check on their mathematical calculations. Sometimes that is intentional, as a teacher decides to simplify and focus on a single part of the cycle for practice or in the interest of time, but that shortfall may also result from a lack of rich modeling problems and pressures to conform to skills stressed in standardized assessments. The next paragraphs present first the perspectives that emphasize the dynamic nature of the modeling cycle, followed by examples of more limited practices.

The Common Core modeling cycle is not meant to imply an order, a linear sequence. As Carol comments, the diagram is accurate, but “this kind of makes it seem static. This is what you would do, but I think often, especially if you’re working with someone else, it’s more dynamic and you can jump all over.”

Denise expressed a similar experience, that in practice there is a back-and-forth flow to the process. “Interpret and validate depend on the subject and depend on the problem. [Students] might find that those two actually come back and forth. So when they try to interpret, they might find, ‘Oh, something’s not right.’ They might want to take it back, revalue, … then they try to interpret again. So it’s a back and forth step.” A one-way recipe is simpler for novices, but students will progress to a more dynamic experience. She says, “I think one-way is more clear for the kids when they first start, but as they start to work on the more complicated modeling, they will find those two actually have to come back and forth.”

School practice is often more limited than what Carol and Denise describe. Authentic problems, adidactical situations, are perhaps rare, and standard problems are commonly used in schools: a certain approach is known to work and students are taught to replicate the standard result. One does not challenge whether the given mathematical model should be applied. Only the student’s application and calculation is validated, and the full cycle never need be repeated. For example, asked whether the Common Core cycle reflected his own students’ steps, Fred summarizes, “I would think of it as they are making a model, they’re doing some computations, they’re interpreting their answers, do they make sense. I do usually do the four steps. They’re not very cyclical though. … It’s usually just once around.” The reason is that the standard approach is expected to work. “I find that it’s almost like the models you use in 9th grade are simple enough. You either can do it and you get the right answer, or you can’t do these steps.”

Ellie, too, states that less experienced modelers assumed a linear process. Students typically think that “you’re supposed to start and do the right work and get the right answer and then you’re done…. Instead, there is a process that actually…. It’s a cycle. You go through it and you come back to the beginning, but it’s not the beginning. You’ve still made progress.” As is discussed previously in the results pertaining to Brousseau's theory, Ellie states that student expectations of a linear process harden into the phenomenon Brousseau calls a didactic contract. They insist that mathematics must work that way. She also links the reluctance to rework results to a broader sentiment. "In psychology we call this 'backup avoidance.' Once you feel like you've made progress it's very hard to give it up, even if you would gain an advantage by going backwards. So it's as true for students as it is for grownups."

Summary of research question #2: the modeling cycle in practice. The subjects of this study are in broad agreement with the Common Core modeling cycle. They use it or similar ideas in their instruction. Their practice, however, is shaped by their students' needs and school realities. On the one hand, teachers add structure and detail to the first stages in the modeling cycle because students have difficulty understanding a problem and structuring a mathematical model. Decomposing steps and scaffolding are more extensive than might be assumed from the cycle diagram. On the other hand, the latter half of the modeling cycle is often treated in a unified way. Interpreting, justifying, and explaining occur together as the student makes the mathematics meaningful in context. Finally, school modeling problems are often subject to standard methods, falling short of novel situations that would require true validation according to real life criteria and that might call for the iterative development of a final solution.

Research Question #3: Instructional Materials

This study's third research question is *What are teachers' perceptions of the appropriateness, ease, and usefulness of mathematical modeling lessons? What additional resources, if any, do teachers report as necessary for the teaching of mathematical modeling?*

Participants' perspectives regarding modeling lessons are consistent with the major themes of their responses to the first two research questions. Materials posing authentic modeling problems are difficult to find or develop. Details of the presentation of the problem are often critical as students sometime lack the real world background knowledge needed to understand the problem situation. Modeling materials that detail and structure the cycle steps are needed, particularly regarding data gathering and organization, decomposing the problem, and the formulate step. Teachers use modeling lessons to develop mathematical content skills, and features that support the alignment with content topics are helpful. In fact, teachers' responsibility to organize their whole curriculum makes a comprehensive set of modeling materials desirable, and lesson collections or repositories receive considerable comment.

Posing problems in context for interest and understanding. The real life context of a modeling problem must be presented in a way that hooks students' interest and gives them insight into the mathematical quantities and relationships that will model the situation. Physically staging the situation can be an efficient way to do that, as, for example, Denise describing herself pointing out the window at the building next door to convey an angle of elevation, or Carol using a student in a roller chair pulled by two ropes. The "Narrow Corridor" problem materials depict a sofa and hallway with drawings and diagrams. Belle is inspired by that lesson (twice she calls it a "great problem"), and she says, "I should scout the school and try to find a corner. Yeah. I think it would help to try to act out the situation, because I can see it from the drawing, but there might be kids who have trouble picturing the situation from the drawing." Belle later says that a small, physical model of the problem would be another technique she might try. "You could actually take cardboard and construct a hallway, and then you could take pencils of different lengths and actually try, you know, try to move them. And see where you get stuck. … It's a physical representation of a real problem." The proper presentation and representation can both engage students and help them understand.

The premise of a sofa in a hallway requires little technical background, but that is not the case for many modeling problems, which require students to absorb unfamiliar subject matter (Burkhardt, 2014; Usiskin, 1997). Fred explains, "It's difficult in practice because … any time you want to do modeling you end up having to sort of explain the context and it could be a lot of reading involved or explaining the problem, and it kind of adds this extra layer of stuff that takes time for the students to figure out." Selecting situations familiar to students may be difficult; students' experiences differ. Fred continues, "There can be a lot of time you have to spend explaining it. If it's a problem about soccer they have to know about soccer. If it's a problem about hockey they've got to know the rules of hockey. And all of that is going to take time out of class. There can be a lot more vocabulary involved. If you're working with ELL students [English Language Learners] or students who don't read that well, you have additional challenges." The constraint is classroom and planning time. "It's kind of a necessary evil. You try to do something fun and interesting, that's relevant to the students' lives, but there is always going to be some kids who probably are not going to have any idea what you're talking about. … you spend a lot of time." Well prepared instructional materials can help give students the necessary background information.

Several subjects mention that video can be an effective way to pose problems (Black et al., 2012). Advantages include the clarity and accessibility of a visual presentation, particularly for weak readers, and a format that some find engages the whole class. The "Narrow Corridor" situation evidently mirrors the storyline of a *Friends* television comedy episode. Alice and Belle both note the connection and say they would use a video clip to introduce the sofa-moving situation and add interest. Alice comments, "I don't really like to show videos unless the video really does directly contribute, and that one [the *Friends* episode] would directly contribute."

Belle draws on students' experience with the television weather news, explaining to them that the "model" that the meteorologist uses as an example of the mathematical modeling the class will be doing. A favorite lesson of Belle's is posed with postal shipping boxes she brings to class. Besides the physical presentation, she animates her explanation with a television reference. "You know that they have that commercial, 'If it fits it ships!' They've all seen that commercial."

Belle also finds that the video format simplifies capture of pertinent quantitative data, a critical step to understand the situation and begin formulating a model. In her class, she uses a video of liquid being poured into a conical martini glass for a related-rates problem. The video substitutes for actual water and glassware, and it conveniently feeds into her measurement software, Logger Pro. Even when the physical phenomenon is not difficult to stage, video is easier to process. For example, in another lesson she captures a bouncing ball on video to be analyzed by software to generate height and time data. Belle's facility with tools that would be more common in a science classroom is perhaps unusual for mathematics teachers, but her examples of modeling situations conveyed in video point toward the potential of instructional media in this format. Belle's efficient capture of quantitative data in electronic format bears on the difficult task of collecting and organizing data, which was discussed in the context of the modeling cycle in research question #2 and returns as a theme for instructional materials.

Flexible materials to teach data collection and organization. The study's subjects have much to say about instructional materials that structure the initial understand-the-problem and formulate phases, helping students organize the salient quantitative factors of a problem (Niss et al., 2007). Two of the study's modeling lessons provide data tables for students to fill in as a first step. How extensive this type of scaffolding should be depends on instructional goals and time constraints. Carol explains how she determines the emphasis and time devoted to collecting and organizing the data to set up a model, “It depends again what your intention is. If you want it to be quick, absolutely it’s quicker to do that [provide a data table]. If you want them to have a relationship with how you find information in the world and this phenomenon in the world, then you may not want to be going for the quicker. You would want to go for that experience. As a teacher you’re always weighing what is my intention here, how much time am I going to give to it, for what purpose.” “I don’t think there is a right or wrong for either. It’s an art. You balance what do my children need, and how much of this can I, how much time can I devote for this based on everything else we’re doing in the year.”

The “Sunrise Sunset” materials include a data table populated with values for students to begin with, but the lesson has several extensions that send the students online for data. The importance, but challenge, of modeling data sourced online is very apparent to teachers. Carol’s students develop a discerning eye for sources of information. She asks them to consider: “Information is ubiquitous, but how do you determine, how do you discern, what’s accurate information and who’s got agenda behind the information, and all of that?” Denise echoes the importance of a critical attitude toward online data. She asks her students, “Where does the information come from? Is it trustworthy? What is the justification based on?” This is part of modeling instruction. She says, “Actually this is part of the learning process about math modeling, they need to learn how to validate their data source.”

Belle notes that the online search in the “Sunrise Sunset” extension is simplified because the materials direct students to a specific website, but “there might be situations where you ask them to get information and there’s just too much out there, and they have trouble distinguishing exactly the piece of information they need.” If we do not expose students to real world complexity and help them develop judgment dealing with it, she says, “We’re not giving them a realistic picture of what things are going to be like. Because when you have a job, and you’re boss says, you know, write me a report on this, you’re going to have to go out and find the information. And it’s not going to be presented to you that way, but as long as you can justify the data that you use, you should be ok.”

Sometimes the information search is straightforward and not worth the overhead of breaking out laptops. Belle says that with respect to the pre-populated data table in the “Sunrise Sunset” basic lesson, “They could have found this on the Internet, but that would take a lot of time.... We’ve got to assume that the kids could look this up for themselves. So that’s good that it is given.” She notes that the clock times provided will need to be converted to decimals. That would be a better use of time. “They have trouble with that, because, you know minutes is not a base ten: it’s a 60 minutes in an hour. So that’s a good mathematical activity.” Alice also approves of the inclusion of these calculation tasks in the materials.

Belle makes a similar evaluation of the “Narrow Corridor” materials, which provide two diagrams and an empty data table that significantly scaffold the model formulation. “I guess that’s why I like when the diagram, the tables, I like when that kind of stuff is provided because that allows more time for the students to do the actual thinking part. ... You know, if the students have to say, ‘How are we going to organize the data? How are we going to do this...’ I mean that takes up a lot of time, whereas they could be taking, you know, if they’re presented with this diagram, then they can use their mental energy to be thinking about the mathematics of it.”

It seems that materials ideally would allow the same lesson to be shaped for different goals and constraints, based on the teacher’s plans. Alice responds that with respect to students collecting data, “I feel like that is an individual circumstances judgment call. I like to think about, what do I want them to get out of it? Do I want them to get out of it the research skill of finding the data on their own, or do I not care about that, and all I really want is for them to see a periodic function in real life?" Scheduling is a primary constraint. She continues, "It kind of comes down to a timing thing. If you have the time to make this a whole, long research project, then, yeah, have them each pick a different city. Then they can compare the sunlight hours of different cities. You can make it a whole, independent data-driven thing. If this is a thing you're doing in a day or two, finding the time of the sunrise and sunset isn't really the main objective." Experience gathering data, Alice says, "definitely matters, but I wouldn't say it matters as much as a lot of the other skills." "I know that I often feel crunched for time. On a long list of standards that need to get covered, going out and finding their own data is not really something that I would prioritize." Denise responds to the question in the same way, "It depends on the time constraint." Flexible materials with features to save time when necessary and offering alternatives to fit the particular needs of a class seem to be called for.

Materials with a step-by-step approach to learning modeling. Data collection and the formulate step bring the most numerous comments from teachers, but there may be a general need for materials configured in modules that can stand alone to fit a particular time slot. Referring to lessons targeting particular modeling cycle steps, Ellie says, "I think that's a good way to get them used to the steps. Again, like I said earlier, one of the major reasons I don't do more modeling is that going through all of the steps in the process, and especially having them re-evaluate and start over completely, takes a lot of time, but if you want them to get there, a good way to do that is to have them do individual steps in the process." Assembling lessons to teach the whole modeling cycle on a step-by-step basis is a considerable effort (Burkhardt, 1989). According to Ellie, "It would be really nice to have some more materials that support those basic steps, in terms of breaking it down. It takes a lot of time to figure out what are the steps that they will need to be able to formulate and solve a problem like this by themselves. And if the Common Core would provide, here are the steps, here are the things you're going to be able to do, and really breaking it down to a basic level." Her request raises the question of how modeling materials fit into the overall curriculum.

Related to mathematics content topics: well designed and well categorized. Teachers must be concerned with the big-picture issues like sequencing, time constraints, preparation, and cultivating a productive classroom culture. They request features in modeling materials to help them plan an integrated course and particularly to weave modeling into the mathematical content sequence. In our interview, Ellie discusses an integrated and sequenced set of modeling lessons stepping through a unit of content instruction, an idea she says she has heard circulate in educational circles. Asked whether such an approach appeals to her, she enthuses, "If I had a unit plan that told me which steps to do where and how to do it? Sure! I would pick that up and go with it. … But the work of figuring out how to do that and what needs to go where, and how you're going to teach all of the topics on this list in nice, neat little modeling units….That would be lovely, but I don't have that, and I'm not really sure how to find problems that would exactly fit everything that I need to teach."

Labeling lessons with an associated mathematical topic is desirable because modeling problems often exercise a particular Common Core content standard, and teachers say a modeling problem is a good way to introduce, assess, or add interest to content skills (Galbraith et al., 2010; Julie, 2002; Niss et al., 2007). Denise explains, "For the teacher, while you're teaching a certain chapter or a certain unit, there is definitely a Common Core standard. If they can search through the standard, and through a link find the math modeling problems, I think that would really help the teacher in their lesson planning, preparations." Even if not for search, administrators often require teachers to tie lessons explicitly to a content standard, a point Denise and Belle each make. As Belle puts it, surveying lesson materials, "It's nice when it has the standards, because lots of times we're required to put the standards on our board."

Fred too views modeling in relation to the content standards, as a teaching resource that fits better with some topics than others (2007). "In the algebra curriculum the modeling goes hand-in-hand with the functions. I find it doesn't go as well with some of the algebraic stuff. It also goes well with the statistics unit. When you're doing modeling with… It's tough to do modeling with polynomials and operations with polynomials and other variables without it feeling really forced." Ellie was more pointed, "The real issue is finding modeling problems that still prepare them for the test they've got to pass. Because unfortunately that is still where the priority is."

Besides addressing the right mathematics content, modeling lesson materials also need to be at the right grade level or difficulty. Alice explains her struggles to adapt Dan Meyer's lesson materials, "I hadn't figured out how to use his problems, because they're pretty involved, and I hadn't figured out a good way to integrate them into my instruction. And so this year was kind of a trial run, 'cause I really like the problems. And I had just had a hard time. Mostly because his, they're at a middle school or Algebra I level, figuring out a good way to integrate problems."

Developing modeling materials oneself is a long process dependent on experimentation (Burkhardt, 1989). As Belle describes, "I've got my 1/3, 1/3, 1/3 rule. One third of the things I try work really well. One third of the things I try don't work at all, and I have to scrap them. And one third of the things I try might work, if I fix them, tweak them a little bit. So, you know, it's taken me seven years to really develop a good collection of things that I know are going to work well, because really only about a third of the things I try work the way I want them to work." It is an inherently exploratory and uncertain design problem, she says. "It's just hit or miss. You just got to try different things."

Some of her lessons Belle has found online, but participants describe the process of finding appropriate modeling materials as haphazard and accidental. Belle explains how a chance contact at a conference turned up a teacher's online lesson database. "She's got week by week, different lessons. …I need something, doing a lesson, or might want something, I'll just look and see if she has anything out there. I assume that's why she put it all out there, for people to share it. Sharing's really good." It is not necessarily materials she authored herself. "She's got this from someplace else. She's just collected it all."

Alice is fortunate that prior teachers at her school left materials on the school server. She is enthusiastic in her description, "It was very well organized in the server…. It was very clear, you know you click into the last year, you click the course you want, you click the unit you want, and the folders are all labeled with each day, and what the topic was that day." In fact, Alice finds the materials shared at her school so useful she imagines a extended system built in the same way. "The server at the school where I work at now, where we can look at things from past years, is by far the most useful source of materials that I have ever used. And I feel like it wouldn't be that hard, to make that bigger, to have it not just be, City Charter School, from past years, but to be some sort of web-based, free mega-server thing. I think that would be incredibly useful. And if you know where to look…. And have it organized in a really clear way. And have it easy for teachers to upload things, so it's basically this massive file sharing." A methodically organized online source of instructional materials with standard features is on several teachers' wish lists. A natural way to organize lesson materials is by the date they are to be taught, or where in the teaching sequence they lie.

The participants say that what is currently available online is sometimes of questionable quality. Denise complains, "Some of the online information, actually the solution is just wrong. We've both had that, and many students and many teachers have had that experience. Because anybody can post an answer online, and nobody really goes online and validates all of the answers. So you have got to find what is a trustworthy web source." Internet search capabilities are obviously available to students as well as teachers, which Denise says complicate and add to her workload. "Before I give out a question, I have to Google online myself and try to make sure the answer is not on the Internet, not the exact same problem. It requires much more work from the teacher's side, because the teacher must take the problem and alter the givens, and alter the problem from the past to make sure the kids wouldn't be able to find the answer online." Ellie also has a great deal of experience with technology. "I think in terms of our students' access to information, an app is the best way to get it to them. Maybe it's something they can log into on Facebook." With advances in technology and student devices, interactive features become a potential component of lesson materials. Ellie cited functions in software that she uses currently for procedural practice as being desirable if implemented for modeling problems as well: automated grading and record keeping.

There is a contradictory element in educators' needs for efficiency versus authentic modeling. Teachers break complex methods into steps and repeatedly expose students to similar problems of increasing difficulty to develop procedural proficiency. In contrast, promoting independent thinking, problem solving, and a productive attitude requires novel problems without familiar, standard solutions. Alice highlights this tradeoff as she catalogues the problems they use at her school, which has a particularly well organized modeling program. The problems reflect the mathematical functions named in the Common Core content standards. “There’s basically four different kinds. The first one is piecewise functions. There’s a distance/time graph and a speed/time graph and they have to talk about the differences between those. Then there is the quadratics, which is either price/profit or projectile. Then there’s exponential functions, which is compound interest or population growth. And then we did systems. Systems were, it was one linear and one either quadratic or exponential. The linear one was you’re shooting an arrow at a bird, and do you hit the bird? And the exponential one was about, oh it was about the sale of iPods versus a made-up fake iPod thing. When do you sell the same numbers of both?” The lack of variety assists, but limits, student learning. She says, “Students know that there are the four types of questions. They can prepare for the four types of questions. And this is the way you do question type two, this is the way you do question type three. I almost think that the prep takes away from the authenticity of it. That they are overprepared for it.”

In the sense of stretching students beyond the application of standard methods, she prefers the “Narrow Corridor” problem, “I think this couch in the hallway question is a better question.” Similarly, the “Sunrise Sunset” lesson is not a conventional application of the sinusoidal function. “I like how it’s a periodic function that’s not the unit circle, and also not a Ferris wheel. I like this. This is something I would use.” Selecting appropriate problems may depend on specific conditions, making an optimal set of modeling problems difficult to widely standardize. For example, note the judgment Carol makes to select her modeling problems. She asks, “Why is it a worthy problem? There’s a lot of talk about what kind of problem would you bring to the students.” “So even though it’s an intellectual exercise, to pick problems that relate to students: A problem that they would really have that would have this similar principle of algebra that they would need to solve it."

Summary of results for research question #3: instructional materials. The participants say that adequate instructional materials are difficult to develop or find. Lessons that pose problems through a physical depiction or with video are often engaging and easy for students to understand. Teachers often modify lessons to fit particular instructional goals, time constraints, and student interests or aptitudes. Such flexibility is facilitated by a modular design breaking the problem into steps and, for the problem and formulate modeling phases, adding scaffolding elements like data input tables, diagrams, and data sets. Modeling is viewed in conjunction with mathematical content areas; therefore, instructional materials should be clearly labeled with applicable content standards, and the ideal collection of modeling problems would integrate with and comprehensively cover the content sequence. Such a set of modeling lessons is not currently available. Teachers rely on their own resources or sets passed down from colleagues.

Research Question #4: Collaboration

This study's third research question is *In what collaborative activities do teachers engage while planning, implementing, and evaluating mathematical modeling lessons? What additional forms of collaboration, if any, do teachers report they would participate in if they were available?*

The first example of collaboration study participants offer is sharing instructional materials. Modeling lessons are long and difficult to develop so a good problem taken from another teacher, particularly with the opportunity to ask questions of the author, is a great aid. Posing the modeling situation to students in an engaging way is a critical skill that merits training and resources like video examples from master teachers. Teachers see the potential of online collaboration, which they say is relatively undeveloped. Teaming with colleagues teaching science is another promising opportunity.

Reducing uncertainty with tested modeling lessons. Modeling is relatively complex and involved, and even an appealing modeling problem is a time commitment. Belle responds to "Narrow Corridor" saying, "I love to do things like this. The main frustration that we've found is the time constraints. I mean this could take a full 90 minute block of time." And a new lesson or problem is always a bit of a risk. Belle says, "You don't want to try something like this just cold on a class without having done it before." Therefore sourcing a proven modeling problem from a teacher who can vouch for it is a great aid. Alice's first thought is, "I think I'd rather take other people's stuff than have long conversations about it." She does see value in an option to contact the author. For example, Alice says she might ask, "Hey, I'm about to do your "Sunrise" activity. Do you have any tips, or what would you change, or how did it go for you?"

The need to collaborate across schools. In fact, Alice has considerable time allocated to working with colleagues. Her charter school administration makes it a priority. "So my school is set up to encourage collaboration. So there is one day each week where an entire department doesn't teach.… It's designed so that we can have a math department meeting, but also so that we can plan with our course partners." She would still like to have more time to work with other teachers, particularly from other schools. Alice says, "It would be nice to collaborate with people outside of my own school building. Like I know that there are excellent school visits that teachers do. And that's nice if you visit on the day when the teacher teaching the class you teach is doing a particularly good lesson. But it would be nice if there were an urban-area Precalculus teachers' meeting, or something, or even like a list serve." The need to go outside the school is to find teachers of the same course, teachers that would use the same problems. As Belle says regarding her school, “Since I’m the only calculus teacher I don’t really have anyone to collaboration with.”

Training to pose the problem effectively. Beyond lesson sharing, teachers also value honing their skills together, mentoring new teachers, and forming a sense of community. A recurring focus is the initial presentation of the problem situation, a difficult and critical step to capture the students’ interest, provide background context, and get them engaged. Ellie, for example, explains her priority for training. “Most of the modeling PDs that I’ve been to has been, ‘Here’s a cool modeling problem. Let’s try to solve it.’ And I feel like that’s fun, but it doesn’t really help support teaching modeling. We’re already math people, we’re already into it, we already want to do that.” Instead, the key is delivery. The teacher is an actor, as Brousseau said. With respect to professional development, “We don’t teach how to engage students.... How do we give this to students so that they’ll be interested in it? The whole idea of modeling problems—ones that have open-ended prompts and open criteria for evaluation—is that any kid should be able to engage with this. So how do we actually entice every kid to engage with this? ... It would be useful to have tips and tricks from people who have already been doing this for years.” Ellie’s opinion is that in-person training is superior to online venues. “Having actual PD where people can raise their hands, share their own anecdotes, and get feedback is more useful. I love educational technology. I think the Internet is certainly the future, but I think that people being together in the same room is still the most valuable way to educate.”

Ongoing support and follow-up is key to collaboration. Ongoing forms of collaboration are more effective. As Carol points out, some educational best practices we apply to children we ignore when it comes to teacher education. “We teach teachers differently than we say you should teach students. You would never go in and teach a concept to kids in one day, and have them figure it out and do it on their own. You would follow up with them. You know what I mean? There's continuity. So I think any kind of in-service change that is made..." should be followed up with ongoing support. "It really works is when there is ongoing mentoring. You know there is that relationship with someone who can help you as you implement it." Carol envisions partnering or mentoring through online communication. In-person meetings and physically visiting classrooms can be supplemented or replaced with technology. "What's often been hard about mentoring is actually finding the time to, for the mentor to be in the classroom. That's when it's hardest for us. But if you use technology, your mentor could be somewhere else, and the mentoring could happen through email, chat, …" or video. For example, reflecting on the "Sunrise Sunset" modeling lesson, she suggests, "If the teacher was trying this, perhaps even a YouTube of a teacher presenting this. 'This is how I would present it to my class.'" Carol directs the new teachers she mentors to watch demonstration videos before launching a new lesson.

In addition to one-to-one mentoring relationships, Carol envisions groups of teachers working together. "You know, there's three other teachers who are doing the same thing, that we can online check in. 'Oh, I figured out that if I did this it really worked well.' 'Oh, I'll try that.'" Ongoing collaboration can grow into a sense of community among a group of teachers embarking on the same set of lessons. "And that brings excitement for the teacher. When you try something new often it can be very lonely. You're kind of like, "Wow. What am I doing here?" But all of a sudden you have this group doing it together, so there's a different… And that goes back to why the students are engaged—and this is part of the Waldorf philosophy—the teacher's excited, and talking to other teachers and saying, 'You know, these three other Waldorf schools are doing the same thing that we're doing right now.' There's a different interest, that we're part of something bigger." The closeness of the Waldorf teachers may be especially strong, but a sense of community is widely felt among educators.

Online communities. Denise thinks in terms of a mathematics community online, an opportunity she says has not been adequately developed. "From my personal experience as a teacher, in the teaching work, the education world we really haven't put technology to its full use…. There's not a single platform for a teacher to communicate their experience teaching math modeling, for example." Denise also teaches computer science, and she says the programming community is much more developed. "I can go and post a question, and other people will respond right away and say, use this example or use this project." Instructional materials are online, but not organized in combination with advice and community support. "There are things like websites where people share their lesson plans. They do exist. They're everywhere." But she wants to be able to ask a question, "to tap in and say, 'I'm teaching topic, and I want to teach this math modeling now, and any ideas, any experience other people have.'" She imagines integrated support, "people's posts, people's blogs, with the teacher's materials, with the real lessons, with everything combined, all available through a very easy to find source, with real live chatting."

Denise is motivated not just to get help, but also to offer it. Experienced teachers want to contribute their advice and the materials they have developed. "Some teachers feel like their voice is not being heard," she says. "They want to share but they do not have the platform to share their experience." Drawing a parallel with pedagogues' faith in student group work, she argues teachers too would benefit from teamwork, not just to be more effective but for professional satisfaction as well. Her enthusiasm is expansive, "Across all the schools, across the country, there is still the need for teachers' communication channel, in collaboration, in teamwork."

Applying mathematical models in collaboration with science teachers. Participants cite collaboration with science teachers as particularly fruitful way to motivate modeling and make it meaningful (2007). For example, Carol says, “Some of the places where it [modeling] best fits in, is we do an interface with the science classes.” Denise agrees, “Working with other math teachers is definitely necessary, but for example physics teachers, chemistry teachers, … There are so many things we can do. Even with the physical education teacher, the gym teacher.” Ellie has the same thought, although she has not implemented it. “The modeling process is something we should talk about with the Design Tech teachers. I’m sure they have a very similar cycle. I don’t know if the wording is the same, but maybe we could work on that.” The application of modeling to science motivates students, and it helps them understand mathematics by giving it context.

Carol coordinates her logarithm instruction with the chemistry teacher’s acidity/bases unit (which employs the logarithmic pH scale), reinforcing application and theory from both subjects’ perspectives. “I set up [logarithms by saying], ‘This is this tool that’s used,’ and then modeling happens a lot in science class. So we have a nice overlap, and we time it that way, so the math teacher can prepare the students from a mathematical/conceptual perspective and to then use the mathematics in their physics or chemistry class.”

Denise sees an opportunity for students to practice organizing data, a priority discussed above. “Because kids collect lab data, they need to analyze lab data. That is preparation for math modeling, for data application. If they start with physics, they see what is really going on in the real world.” In physics class students learn, “how the parabola is applied in real life.” She cites the challenge of teaching students estimation as more naturally tackled in collaboration with science teachers, and Denise also emphasizes engagement benefits (generating student excitement is a repeated theme for Denise), “They would say, ‘Hey we learned this! This is some of my junior and senior kids. “‘Oh we learned this in physics.’ Or ‘We learned this in Chemistry.’ Or some of the kids, the biology teacher taught us. So you can tell estimation is more from science than from math, because in science they are dealing with real life.”

Denise laments, however, that institutional structures prevent her from collaborating with science colleagues as much as she would like. Reflecting on her own education—Denise grew up in China—she says they teach “physics and geometry side by side. So the math teacher will work very closely with the science teacher, especially the physics teacher, or chemistry teacher. They will work together and design some problems for the kids to learn both subjects at the same time.” But this is not possible here. “I find that, first of all, it’s not allowed by the curriculum. Our curriculum design doesn’t allow teachers to have those kinds of flexibilities. Second, I find that there is a lack of coordination among all of the departments, even though in our school our administration really encourages the teachers to have the cross-department experience, but because of the schedule, because of the school schedule, it became really difficult.”

Summary of results for research question #4: collaboration. Sharing instructional materials is the first example of collaboration teachers cite, but, because within a school they are frequently the only one teaching a specific course, they often must look outside. That suggests online collaboration, which teachers believe has potential, but is not currently well developed. While teachers say the chance to watch a video of a master teacher introducing a modeling lesson or to ask a question regarding a problem are beneficial, they also want richer forms of collaboration. In-person training, mentorships, and ongoing relationships improve practice over time, and they also offer a sense of professional community that is valuable.

Chapter V: Summary, Conclusions, and Recommendations

Summary

The purpose of this multicase study was to further understanding of the implementation of the Common Core modeling standard. The study proceeds by examining teachers' perspectives of the modeling cycle—or modeling process—specified by the Common Core, modeling instructional materials, and collaborative activities. Teachers are the focus of the research because they are authorities on the roll out of the Common Core in the dual sense that they are well informed about the practical issues of what does and does not work in their classrooms, and they are the decision makers regarding how and whether mathematical modeling is taught, what materials they use, and who they work with. A theoretical framework developed by applying Guy Brousseau's pedagogical theories to the modeling literature guided the study. Brousseau is a prominent French educator and researcher. His theory of didactical situations in mathematics is a comprehensive, detailed description and guide for mathematics education.

The analysis of participant interview data showed that teachers' views and practices align with the Common Core modeling standard and that they structure their instruction in a way that is consistent with it and the modeling research upon which the standard is based. Shifting to the teacher's vantage, as this study has done, also uncovered important additional dimensions regarding how mathematical modeling is and should be taught, additional detail that can help curriculum developers, teacher trainers, administrators, and researchers better support teachers as they implement the modeling standard. This chapter begins with an explanation of such conclusions and then follows with a set of recommendations.

Conclusions

A theme running through all of the findings is that teachers must integrate a multitude of concerns as they plan and execute their responsibilities: holding students' attention and motivating them, helping them develop mathematical understanding, following a sequence that incrementally develops the overall curriculum, assessing the students, producing high marks on standardized tests, and fitting this all into 45-minute (or some standard length) periods. Mathematical modeling actually fits very well with parts of this agenda, particularly student engagement and meaning making (conceptual understanding); however, in many cases support for teachers that desire to employ modeling is unsatisfactory in critical respects. As a practical matter, instructional support such as a published modeling lesson or a modeling workshop that ignores or contradicts teachers' classroom constraints will have limited impact. For example, a lesson running through the whole modeling cycle that requires three class periods to complete will rarely be used; nor will research quantifying learning gains promote modeling with the same urgency as would evidence of high-stakes test impact. Several findings of this study concern how addressing teacher requirements can significantly enhance the usefulness of modeling materials. The first example is discussed in the next section: how modeling problems' ability to give context to mathematical ideas and thus give new concepts meaning can serve course objectives if lesson materials are organized for use within the context of the content sequence.

Meaning making in context: modeling as vehicle. Teachers value real world applications of mathematics because they engage and motivate students, and because the students' experience with the problem context helps them make sense of mathematical concepts. Modeling researchers call this viewpoint "modeling as vehicle." Brousseau put problem context at the center of learning, and he took pains to creatively configure situations to stimulate meaning making and to sequence lessons intentionally. Teachers desire to use modeling problems in this way, and sometimes they do, but it is more difficult than the traditional practice of teaching an isolated mathematical procedure and then practicing it with word problems. The study uncovered several organizational and design practices that help teachers to employ modeling problems as a vehicle to teach mathematics content.

First, teachers select modeling lesson materials to address content standards. Organizing published lessons with clear identification and in chronological order would make it easier for teachers to select them at the appropriate point in the curriculum. Second, isolating the steps in the modeling cycle that develop conceptual understanding would help teachers focus on that use, perhaps trimming other parts of the cycle given time constraints. Third, teachers describe their most effective modeling lessons as requiring real world background that the students are already familiar with, and they often use physical models or props to help students make sense of the situation. Materials, training, and supports that coordinate the complementary goals of content standards and modeling should lead to greater adoption and success.

Posing a problem to students is a key step. Brousseau coined the term “devolution” to encompass both introducing the situation and establishing agency in the student’s mind. Teachers enthusiastically recount their technique and creativity initiating a successful problem, but cite inexperience presenting a lesson for the first time as a hurdle. Support that addresses the problem-posing step is in demand: written instructions and video examples, training, mentorship or the opportunity to ask questions of someone familiar with a particular problem, and resources like video introductions that lighten the demands on teachers.

The modeling cycle as a disassembly guide: focus & time. The participants agreed that the modeling cycle accurately depicts the steps modelers follow and the process that teachers desire students to learn. To teach modeling itself, modeling as content, teachers parse the process and focus on a particular phase, step, or part of a step for repeated practice and development. For example, they may have students practice the three modeling steps interpret, validate, and report using a given algebraic functional model in varying real world contexts. These three steps composing the latter half of the modeling cycle are often treated holistically. In contrast, when teaching the first half of the modeling cycle, they sometimes focus on single modeling step, or a part of one. For example, collecting and formatting raw data in a table is a skill useful for the understand-the-problem step. Working with estimates and making the critical decision to rework a model that falls short in the validation step are other difficult skills that teachers identify for focused instruction. While it is true that an assignment to complete the whole cycle is sometimes given, for example as a summative chapter assessment, a specialized and focused lesson is more common. Unfortunately, instructional materials explicitly designed to address such an incremental approach are not as well developed as are examples of complete modeling tasks. A gap exists: sequences of problems designed to incrementally develop modeling competencies in a focused time period. Professional development has a similar over-focus on working complete problems, as opposed to, for example, training for the critical step of posing the problem situation to students. The modeling cycle is a road map for the overall modeling process, but it can also serve as a guide to develop and organize materials that address individual steps.

Available modeling materials lack flexibility and creative delivery. When the study's subjects examined published modeling problems, they made detailed comments about each feature, considering how it might suit a particular lesson goal or how it might be adapted to suit different types of students. Their attention was especially focused on the manner the situation was presented, explained, and scaffolded to help students though the understand-the-problem and formulate-the-mathematical-model steps. They suggested a wide variety of diagrams, physical models, input guides, online information sources, data tables, video presentations, instrumentation, commercial contexts, and related supports that they identified as valuable or suggested as additions. A complete collection of instructional materials that could be used to present a modeling situation goes well beyond a linear presentation in a textbook or modeling handbook. Generally, teachers compose or assemble these components themselves, a time consuming process (and one requiring skill and experience) that could partially be reduced if modeling problems included a much more varied and creative set of supplemental materials. Particularly since teachers say they take many of their modeling lessons from Internet sources, such a compilation might be practically delivered online. Related professional supports that the participants recommended—teaching notes, videos of veteran teachers presenting the problem, opportunities to ask questions—could be integrated with the lesson materials.

The organization and generation of instructional materials. Study participants describe the sources of their modeling problems as personal and ad hoc. They take a lesson from a colleague or find one online that meets a requirement in their course, adapt it, and refine it over time. Viewed from the perspective of a teacher who must organize daily lessons to cover a course curriculum, the ideal instructional materials are not just stand-alone modeling lessons; they are a collection of problems that are sequenced and integrated with the other components of a particular course. The sources of materials that the subjects described as best organized were complete and chronological directories of lessons that had been developed and assembled by other teachers covering similar courses. While published modeling materials do have some features to identify when they might be used, a content standard label for example, they are not organized as comprehensive sources. Participants imagined and suggested the development of an extensive repository of lessons, and they referenced the popularity of teachers sharing their materials as suggesting the plausibility that such a large body of work could be donated by teachers.

The participants collaborate with colleagues to develop modeling lesson materials and, to a lesser extent, to discuss how to present problem situations. There is a particular difficulty connecting teachers who teach the same subject (i.e. Algebra 1, or Geometry), because in smaller schools only one teacher may cover a particular course. This highlights the specificity of both pedagogical content knowledge and instructional materials. It also suggests that, as in the case of online lesson repositories, communication technology may provide a platform for collaboration among teachers with similar classes but different locations.

Recommendations

Reflecting on the path this research took over several years, a key point was the selection of Guy Brousseau's theory of didactical situations as the conceptual framework. From that point, reviewing the literature became more focused and interesting, and developing the methodology became more productive, particularly preparing for the interviews and, later, data analysis. I recommend doctoral students select and explore their study's theoretical framework early in their research. Choosing an area "off the beaten path," as I have done with Brousseau (at least to the English-speaking world), has been especially interesting and rewarding.

The findings of the study suggest several areas of research and development that would be important to teachers and to their implementation of modeling in classrooms. Research that measures the effectiveness of modeling instruction in terms of student performance on standardized tests would be decisive to motivate teachers and administrators. Similarly, research measuring actual classroom time constraints would address a second major constraint for teachers. For example, if video presentation of a problem saved time, or if written, typed, or oral interpretation of modeling results was found to be more efficient, those results would help drive the development of superior materials and practices. Research questions such as these could be studied using strictly quantitative methods, but qualitative data—the perspectives and interpretations of teachers and other practitioners—help us understand the meaning and importance of quantitative results. I recommend aspiring researchers obtain a solid background in qualitative methods and employ them in their studies.

Online repositories of instructional materials and pedagogical support hold obvious promise, but, equally, have not yet been successful. The complex interplay of factors such as sourcing the materials, organizing them, curation, differentiation, legal ownership, and commercial model call for an engineering approach to research and development. The findings of this study suggest that organizing by course using a chronological metaphor would be appealing to teachers. Precedents in software development suggest that different online legal and commercial models can coexist. Perhaps analogs from the software industry can be adapted directly for education. In particular, I recommend an online community basis for the development of modeling materials and professional support, perhaps sponsored by the NCTM. A wellspring of desire to help other teachers and a pride in one's expertise and curricula could be a potent generative force if there were a structure and venue for contributions.

References

Bell, M. S. (1971). Mathematical models and applications as an integral part of a high school algebra class. *The Mathematics Teacher*, 293-300.

Black, P., Burkhardt, H., Daro, P., Jones, I., Lappan, G., Pead, D., & Stephens, M. (2012). High-stakes examinations to support policy: Design, development and implementation. *Educational Designer*, *2*(5). Retrieved from http://www.educationaldesigner.org/ed/volume2/issue5/article16/pdf/ed_2_5_ISDDE_12.pdf

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and Its Applications*, *22*(3), 123-139.

Bloomberg, L. D., & Volpe, M. (2012). *Completing your qualitative dissertation: A road map from beginning to end* (2nd ed.). Thousand Oaks, CA: Sage.

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application*, *1*(1), 45-58.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects — state, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37-68. doi:10.1007/BF00302716

Blum, W., Niss, M., & Galbraith, P. (2007). Introduction. In *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 3-32). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM*, *38*(2), 86-95.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51-63.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Brousseau, G. (1999). Research in mathematics education: Observation and… mathematics. In *European research in mathematics education I* (Vol. 1, p. 34).

Brousseau, G. (2006). Mathematics, didactical engineering and observation. In *PME Conference* (Vol. 30, pp. 1-16). Retrieved from http://www.math.washington.edu/~warfield/PME-30 Plenary Address.pdf

Brousseau, G., Brousseau, N., & Warfield, V. (2013). *Teaching fractions through situations: A fundamental experiment.* New York, NY: Springer.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Burkhardt, H. (1983). *An international review of applications in school mathematics—the elusive El Dorado*. Retrieved from http://files.eric.ed.gov/fulltext/ED230443.pdf

Burkhardt, H. (1989). Modelling and curriculum change. *Teaching Mathematics and Its Applications*, *8*(4), 158-161.

Burkhardt, H. (2014). Curriculum design and systemic change. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 13-34). doi:10.1007/978-94-007-7560-2

Burkhardt, H., & Pollak, H. (2006). Modelling in mathematics classrooms: Reflections on past developments and the future. *ZDM*, *38*(2), 178-195.

Burkhardt, H., & Schoenfeld, A. H. (2003). Improving educational research: Toward a more useful, more influential, and better-funded enterprise. *Educational Researcher*, *32*(9), 3-14.

Charmaz, K. (2014). *Constructing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed.). Thousand Oaks, CA: Sage.

DePeau, L. C., III. (2012). Sunrise sunset. In H. Gould, D. R. Murray, & A. Sanfratello (Eds.), *Teachers College, Columbia University mathematical modeling handbook* (pp. 175-182). Bedford, MA: Consortium for Mathematics and Its Applications.

Fey, J., Garfunkel, S., Briars, D., Isaacs, A., Pollak, H., Robinson, E., Scheaffer, R., Schoenfeld, A., Seeley, C., Teague, D., & Usiskin, S. (2014). The future of high school mathematics. *Mathematics Teacher*, *107*(7), 488-490.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, *1*(1-2), 3-8. doi:10.1007/BF00426224

Galbraith, P. (2007). Authenticity and goals—overview. In *Modelling and applications in mathematics education* (pp. 181-184). New York, NY: Springer.

Galbraith, P. L., Stillman, G., & Brown, J. (2010). Turning ideas into modeling problems. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 133-144). New York, NY: Springer.

Garfunkel, S. A., & Malkevitch, J. (1994). *For all practical purposes: Introduction to contemporary mathematics.* New York, NY: WH Freeman.

Glaser, B. G. (1978). *Theoretical sensitivity: Advances in the methodology of grounded theory.* Mill Valley, CA: Sociology Press.

Glaser, B. G., & Strauss, A. L. (2009). *The discovery of grounded theory: Strategies for qualitative research.* New Brunswick, NJ: Transaction Publishers.

Gould, H. (2013). *Teachers' conceptions of mathematical modeling* (Doctoral dissertation). Available from ProQuest Dissertations and Theses Global. (UMI No. 3560822)

Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist*, *53*(1), 5.

Herbst, P. G. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education*, 313-347.

Huson, C. J. (2015). Employing real-world resources to teach mathematical modeling. In *Teachers College, Columbia University mathematical modeling handbook III: Lesson paradigms.* Bedford, MA: Consortium for Mathematics and its Applications.

Julie, C. (2002). Making relevance relevant in mathematics teacher education. In *Proceedings of the 2nd international conference on the teaching of mathematics (at the undergraduate level).*

Kaiser, G., Lederich, C., & Rau, V. (2010). Theoretical approaches and examples for modelling in mathematics education. In B. Kaur & J. Dindyal (Eds.), *Mathematical applications and modelling: Yearbook 2010, association of mathematics educators* (pp. 219-246). Hackensack, NJ: World Scientific.

Kent, M. R. (2010). *An analysis of Brousseau's milieu in a community college's calculus laboratory* (Doctoral dissertation). Available from ProQuest Dissertations and Theses Global. (UMI No. 3447941)

Krulik, S., & Reys, R. E. (1980). *Problem solving in school mathematics* (Vol. 28). Reston, VA: National Council of Teachers of Mathematics.

Legé, G. F. (2003). *A comparative case study of contrasting instructional approaches applied to the introduction of mathematical modeling* (Doctoral dissertation). Available from ProQuest Dissertations and Theses Global. (UMI No. 3091273)

Lesh, R., Doerr, H., Carmona, & G., Hjalmarson, M. (2003). Beyond constructivism. *Mathematical Thinking and Learning*, *5*(2), 211-233. Retrieved from http://ejournals.ebsco.com/direct.asp?ArticleID=JHBC2TNMP9M1PWYL6HE7

Lesh, R., & Yoon, C. (2007). What is distinctive in (our views about) models & modelling perspectives on mathematics problem solving, learning, and teaching? In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 161-170). New York: Springer.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education*, 660-675.

Maxwell, J. A. (2012). Conceptual framework. In *Qualitative research design: An interactive approach.* Thousand Oaks, CA: Sage.

McShane Warfield, V. (2014). *Invitation to didactique.* New York: Springer.

Merriam, S. B. (1998). *Qualitative research and case study applications in education.* San Francisco, CA: Jossey-Bass.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association and the Council of Chief State School Officers (NGA & CCSSO). (2010). Common core state standards for mathematics. Retrieved from http://www.corestandards.org/

Niss, M. (2012). Models and modelling in mathematics education. *European Mathematical Society*. Newsletter, (86), 49-52. Retrieved from http://www.ems-ph.org/journals/newsletter/pdf/2012-12-86.pdf

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.), *ICMI Study Series: Modelling and applications in mathematics education* (Vol. 10). New York, NY: Springer.

Pollak, H. O. (1968). On some of the problems of teaching applications of mathematics. *Educational Studies in Mathematics*, *1*(1), 24-30.

Pollak, H. O. (1969). How can we teach applications of mathematics? *Educational Studies in Mathematics*, *2*(2), 393-404.

Richards, L., & Morse, J. M. (2013). *Readme first for a user's guide to qualitative methods* (3rd ed.). Thousand Oaks, CA: Sage.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York, NY: Macmillan.

Schoenfeld, A. H. (2012). Problematizing the didactic triangle. *ZDM*, *44*(5), 587-599.

Schoenfeld, A. H. (2013). Mathematical modeling, sense making, and the Common Core State Standards. *Journal of Mathematics Education at Teachers College*.

Schoenfeld, A. H. (2014). Reflections on curricular change. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 49-72). doi:10.1007/978-94-007-7560-2

Strauss, A., & Corbin, J. M. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Thousand Oaks, CA: Sage.

Tan, L. C. (2012). Narrow corridor. In H. Gould, D. R. Murray, & A. Sanfratello (Eds.), *Teachers College, Columbia University mathematical modeling handbook* (pp. 67-76). Bedford, MA: Consortium for Mathematics and Its Applications.

Usiskin, Z. (1997). Applications in the secondary school mathematics curriculum: A generation of change. *American Journal of Education*, 62-84.

Vialva, J., & Huson, C. (2013). Assessing "Gauging Rainfall." In N. Fletcher, A. Velamur, B. Waid, & A. Dimacali (Eds.), *Teachers College, Columbia University mathematical modeling handbook II: The assessments* (pp. 109-122). Bedford, MA: Consortium for Mathematics and Its Applications.

Vos, P. (2011). What is 'authentic' in the teaching and learning of mathematical modelling? In G. Kaiser (Ed.), *Trends in teaching and learning of mathematical modelling* (pp. 713-722). New York, NY: Springer.

Yin, R. K. (2009). *Case study research: Design and methods* (4th ed.). Thousand Oaks, CA: Sage.

Zawojewski, J. (2010). Problem solving versus modeling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 237-243). New York, NY: Springer.

Appendix A

Intake Form

Interview Data Template

Subject Name:

Date:

School where currently teaching:

Years of experience teaching mathematics:

Courses taught (grade):

General teaching practices

Where do you get the instructional materials for a typical lesson? Give the approximate percentage from each source:

Textbook ____ Websites ____ Write your own ____ Other ____

(name sites)

How often do you plan your lessons with someone else? (e.g., half the lessons, once a week, once a month)

Modeling

Describe your own experience, if any, with mathematical modeling in your high school or college studies.

Describe how familiar you are with the CCSSM modeling standard:

1. Not very familiar: heard of it but not much more

2. Somewhat: read about it

3. Considerable: thought about it in terms of my own teaching

4. Experienced: teach modeling to meet the standard, or taken PD on the topic

Describe any modeling lessons that you have taught.

Appendix B

Interview Protocol

Intake Form: Consent

Thank you for talking with me. This research is for my dissertation at Teachers College, Columbia University. The purpose is to study teachers' perspectives on teaching mathematical modeling. I'm not evaluating your answers, I'm listening to your opinions for ideas to help make teaching modeling easier and better.

I'll use a pseudonym, not your name, so your identity won't be public in my dissertation.

There are two forms to start with, the Informed Consent form and the Participant's Rights form. Please read them and ask any questions you have. If it is ok for me to tape the interview please check that space, and if you agree then sign and date the form.

[Verbally summarize Informed Consent & Participant's Rights forms; 2 copies]

Introduction: RQs

By modeling I mean teaching students to apply mathematics to real-world problems, specifically the the Common Core definition.

One of my research questions is what teachers think of the C-Core modeling cycle.

Another is where you get modeling lessons, and how those could be improved

Finally, do you collaborate when you plan modeling, and would it help if there was more of a chance to work with other teacher.

So it's all about what could help teachers: pedagogy, instructional materials, collaboration. How good is it now, what's important, how could it be better.

Intake Forms Copy

I'd like to start with your teaching background, using this template, which I'll fill in as you talk.

[Complete 1st section / Interview Data Template]

Can we start with just a very open-ended question: Tell me your general perspective on modeling, if you have one.

The Modeling Cycle

The Common Core uses this modeling cycle (graphic) to show the steps modelers follow (not necessarily in order, and not only a single time through)

[Common Core modeling cycle and standards document]

How, if at all, does this relate to how you structure your modeling instruction? (And assessment)

Which steps are the most challenging for students? (to the extent it is an appropriate framework)

Is instruction structured to target modeling competencies, or to target mathematical concepts?

Meaning Making, Devolution; Authenticity

Thinking in terms of what you would consider "good" modeling problems, what about the problems make them "good"?

[Motivating, engaging, empowering students to think independently, "authentic", socially activating, affecting how the understand and view the world, etc.]

Expanded Milieu

Real world problems may not be nice and neat. Information may be missing. Students may lack background. Talk about the scope of students' actions to understand the problem: group work, classroom resources, internet, etc.

Talk about the skills and tools necessary to learn about and understand modeling problems.

Talk about whether and how modeling lessons address this.

Didactic Contract

Estimating values and using appropriate precision is sometimes new to students modeling. Do you have experiences or perspectives in this regard?

Here is a task from a modeling course given at Teachers College, "How big is the bail of hay?" with no given values. How would your students react? Discuss the value of this modeling problem.

[Review and discuss modeling lesson material]

Scaffolding Mathematizing

I have two lessons to show for your perspective. First discuss your instructional materials: sources, quality, adequacy, priorities, requests and suggestions.

[Narrow corridor & Sunrise sunset]

The mathematizing step is difficult and sometimes time consuming. Should it be scaffolded and/or differentiated? How?

From the perspective of making life easier for the teacher, what is important regarding instructional materials?

Collaboration

Discuss if and how you work with other teachers, how that could be improved and how important that would be.

Appendix C

“Bale of Straw” Modeling Task

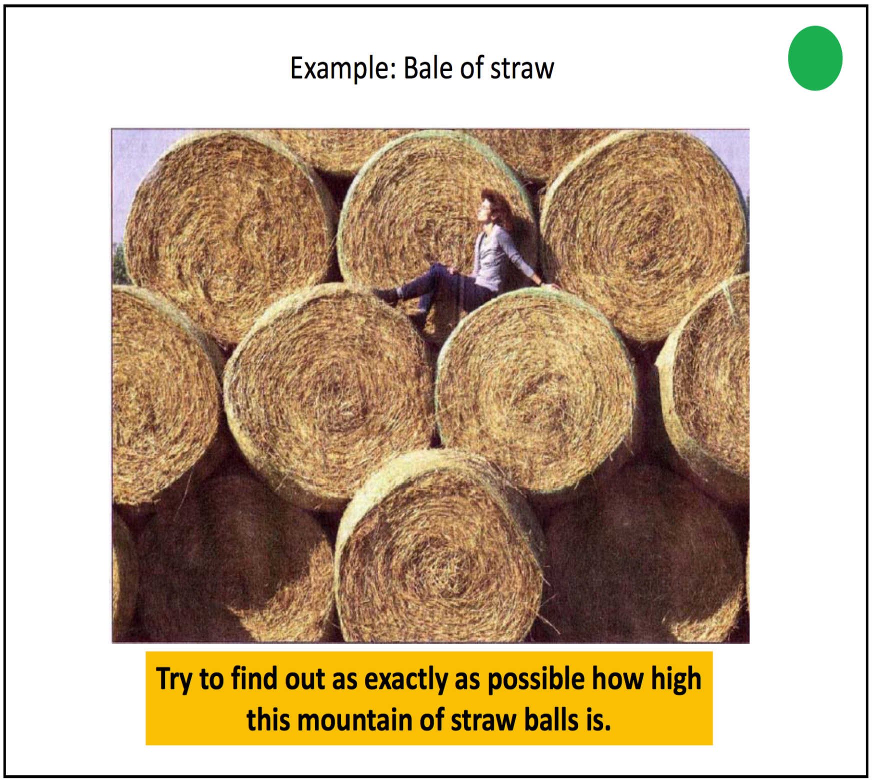

Figure A1: The “Bale of Straw” modeling task was used in interviews to elicit perspectives on a loosely structured, open problem situation without specific given quantities (Kaiser et al., 2010).